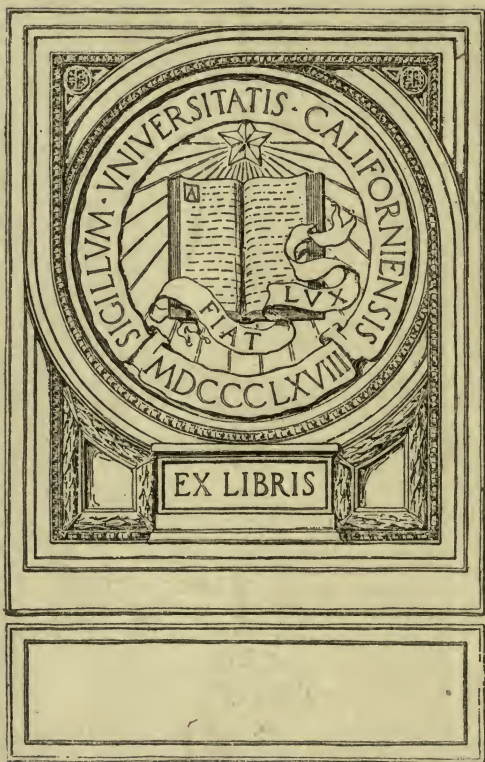


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SOLUTIONS OF EXAMPLES  
IN  
CONIC SECTIONS,

TREATED GEOMETRICALLY

BY

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THIRD EDITION, REVISED.



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## PREFACE TO SOLUTIONS.

I HAVE frequently received requests for a book of Solutions of the Examples in my treatise on Conic Sections, but have never been able to find time to prepare them.

Mr Archer Green, B.A., Scholar of Christ's College, volunteered to undertake the task, with the aid of my notes and his own, and, with the exception of a few at the end, wrote out the solutions entirely.

Mr Green was however prevented by illness from completing the revision of the proofs, and I am much indebted to Mr J. Greaves, Fellow of Christ's College, who kindly undertook to examine the rest of the sheets.

The book will, I hope, prove useful both to students and teachers, as a companion volume to the treatise on Conic Sections.

W. H. BESANT.

*Sept.* 1881.

## PREFACE TO THE THIRD EDITION.

THE solutions have been revised, and many additions have been made to them. They will now be found to be in complete accordance with the sixth edition of the Geometrical Conics.

W. H. BESANT.

*Jan.* 1890.

# CONIC SECTIONS.

## SOLUTIONS OF EXAMPLES.

### CHAPTER I.

1. IF the tangent at  $P$  meet the directrix in  $Z$ , and  $S$  be the focus,  $PSZ$  is a right angle;  
 $\therefore S$  lies on the circle of which  $PZ$  is diameter.

2. Let  $PN$  and  $QM$  be the ordinates at  $P$  and  $Q$ .  
 Then  $PN : QM :: SP : SQ :: XN : XM$ ;  
 $\therefore$  the triangles  $PXN$  and  $QXM$  are similar and  $PX$ ,  $QX$  equally inclined to  $XS$ .

3. By Art. 8,  $FS$  is the external bisector of the angle  $PSQ$ .

4.  $SP : PK :: SA : AX :: SE : EK$ ;  
 $\therefore EP$  bisects the angle  $SPK$ .

5. Since  $F$ ,  $S$ ,  $P$  and  $K$  lie on a circle,  
 the angle  $KSF$  = the angle  $FPK$  = the angle  $FTS$ .

6.  $PN : P'N' :: SP : SP'$ ;  
 $\therefore XK : XN :: XK' : XN'$ ;  
 $\therefore$  the angle  $LNN'$  = the angle  $K'N'X$  = the angle  $LN'N$ .

7. Let  $Q$  be the point where the tangent at  $R$  meets  $NP$ .

Then  $NQ : NX :: SR : SX :: SA : AX :: SP : NX$ ;  
 $\therefore SP = QN$ .

8. Let  $SY$  be perpendicular to the tangent at  $P$  and  $GL$  perpendicular to  $SP$ .

Then, since the triangles  $PSY$ ,  $GPL$  are similar,

$$PG : PL :: SP : SY,$$

or

$$PG : SR :: SP : SY.$$

9. If the tangent meet the directrix in  $Z$ , and  $SP$  be drawn such that  $ZSP$  is a right angle meeting the tangent in  $P$ ,

then  $P$  will be the point of contact of the tangent  $ZP$ .

10. If  $P$ ,  $Q$  be the extremities of the chord, and  $PK$ ,  $QL$  be perpendicular to the directrix,

$$SP : PK :: SA : AX :: SQ : QL;$$

$$\therefore SP + SQ :: PK + QL :: SA : AX.$$

Now the distance of the middle point of  $PQ$  from the directrix is equal to half  $PK + QL$ , and is therefore least when  $SP + SQ$  is least, that is, when  $PQ$  goes through the focus.

11. If  $TP$ ,  $TP'$  be the fixed tangents, and the tangent at  $Q$  meet them in  $E$ ,  $E'$ ,

the angle  $PSE$  = the angle  $ESQ$ , and the angle  $QSE'$  = the angle  $E'SP'$ ;

$$\therefore \text{the angle } ESE' = \text{half the angle } PSP'.$$

12. If perpendiculars from the given points  $PK$ ,  $QL$  be drawn to the directrix and  $S$  be the focus,

$$SP : SQ :: PK : QL, \text{ a constant ratio;}$$

$\therefore$  the locus of  $S$  is a circle.

13. Let the normal at  $P$  meet the axis in  $G$ .

Taking  $O$  as the fixed point in the axis, it is obvious that the triangles  $OSR$ ,  $GSP$  are similar;

$$\therefore SO : SR :: SG : SP :: SA : AX;$$

$\therefore SR$  is constant, and  $R$  lies on a circle of which  $S$  is the centre.

$$14. \quad AT : AX :: SR : SX :: SA : AX;$$

$$\therefore AT = AS.$$

15.  $ST$  bisects the angle between  $SP$  and  $SQ$ , Art. 12, and  $SR$  bisects the angle between  $QS$ , and  $SP$  produced, Prop. II., Art. 5;

$\therefore RST$  is a right angle.

16. The triangles  $EAT$ ,  $ERS$  are similar;

$$\therefore AT : SR :: EA : ER :: AX : SX;$$

$$\therefore AT : AX :: SR : SX :: SA : AX;$$

$$\therefore AT = AS.$$

17. If  $TL$  be perpendicular to the directrix,

$$SR : TL :: SA : AX :: SM : TL;$$

$$\therefore SM = SR.$$

18.  $FS$  is the external bisector of the angle  $QSP$ , and  $F'S$  of  $QSP'$ ;

$$\therefore \text{the angle } FSF' = \text{half the angle } PSP'.$$

19. Since the triangles  $SPN$ ,  $SGL$  are similar,

$$\therefore GL : PN :: SG : SP :: SA : AX.$$

20. If the normals  $PG$ ,  $P'G'$  meet in  $Q$ , and  $QV$  be drawn parallel to the axis to meet the chord in  $V$ ,

$$VQ : VP :: SG : SP :: SA : AX :: SG' : SP' :: VQ : VP';$$

$$\therefore VP = VP', \text{ or } V \text{ bisects } PP'.$$



21.  $DS$  is the external bisector of the angle  $PSQ$ , and  $ES$  of  $pSQ$ ;

$\therefore DSE$  is a right angle.

22. The semi-latus rectum is an harmonic mean between  $SP$  and  $SP'$ ;

$$\therefore 2SP \cdot SP' = SR \cdot PP'.$$

23.  $PE : PL :: PQ : PG :: PV : PS :: PP' : 2SP$ ,  
see Ex. 20 ;

$$\therefore PE : SR :: SP' : SR;$$

$$\therefore PE = SP'.$$

Similarly

$$P'E = SP.$$

24. The right-angled triangles  $DSQ$ ,  $DSE$  have a common hypotenuse.

Also  $SE = SR = SQ$ ;

$\therefore$  the angle  $QSE =$  the angle  $ESP$ .

25. Let  $S$  be the focus and  $P$  and  $Q$  the given points. Through  $P$  draw a straight line  $PK$  so that  $SP$  may bear to  $PK$  the given ratio of the eccentricity.

Through  $Q$  draw a straight line  $QL$  so that  $SQ : QL$  in the same ratio.

With centres  $P$ ,  $Q$  and radii  $PK$ ,  $QL$  respectively describe circles.

The perpendicular from  $S$  on a common tangent to these circles will be axis.

26. Let the tangents at  $P$  and  $Q$  intersect in  $T$ .

Draw  $TN$  perpendicular to directrix and  $TM$  perpendicular to  $SP$ .

Then  $SM : TN :: SA : AX$ .

But  $ST$  bears a constant ratio to  $SM$ , since angle  $TSM =$  half  $PSQ$  ;

$\therefore ST$  bears a constant ratio to  $TN$ .

27. Let  $T$  be the intersection of the tangents at  $P$  and  $p$ .

Draw  $TK$  perpendicular to  $Pp$ .

Then  $TK : PL :: TP : PG$  and  $TK : pl :: Tp : pg$ .

Again, draw  $GM, gm$  perpendicular to  $SP, Sp$  respectively,  
and  $TN, Tn$  perpendicular to  $SP, Sp$  respectively.

Then  $TP : PG :: TN : MP :: Tn : mp :: Tp : pg$ ;

$$\therefore TK : PL :: TK : pl ;$$

$$\therefore PL = pl.$$

## CHAPTER II.

### THE PARABOLA.

1. THE distance of the centre of the circle from the fixed point is equal to its distance from the fixed straight line, and therefore its locus is a parabola of which the fixed point is focus and the fixed straight line directrix.

2. Through the vertex draw a straight line making the given angle with the axis; the tangent at the point where the diameter bisecting this chord meets the curve will be the tangent required.

Or, draw a radius vector from the focus, making twice the given angle with the axis.

3. Since  $TA = AN$ ,  $PN = 2AY$ ;  $\therefore AY^2 = AS \cdot AN$ .

4. Let  $SY'$  be drawn perpendicular to the line through  $G$  parallel to the tangent.

Then in the right-angled triangles  $YST$ ,  $Y'SG$ ,  $ST = SG$ , and the angles  $YST$ ,  $Y'SG$  are equal;

$$\therefore SY = SY'.$$

5. Draw  $SY$  perpendicular to the tangent and  $YA$  perpendicular to the axis.

Produce  $SA$  to  $X$ , making  $AX$  equal to  $SA$ .

Then the straight line through  $X$  perpendicular to  $SX$  is the directrix.

6. Let the circle touch the fixed circle in  $Q$ , and the straight line in  $R$ ; let  $P$  be its centre, and  $S$  the centre of the fixed circle.

Produce  $PR$  to  $M$ , making  $RM$  equal to  $SQ$ , then the

straight line  $MX$  drawn through  $M$  parallel to the given line is a fixed straight line.

Then, since  $SP$  is equal to  $PM$ , the locus of  $P$  is a parabola of which  $S$  is focus and  $MX$  directrix.

7. Draw  $SY$ ,  $SY'$  perpendiculars on the two tangents.

Then, if  $SA$  be perpendicular to  $YY'$ ,  $A$  is the vertex.

Produce  $SA$  to  $X$ , making  $AX$  equal to  $AS$ ;  $X$  is the foot of the directrix.

8. If the tangent at the end of the latus rectum meet  $PN$  in  $Q$ ,

$$QN = XN = SP.$$

9. Since  $SYP$  and  $PNS$  are right angles,  $P$ ,  $N$ ,  $S$ ,  $Y$  lie on a circle ;

$$\therefore TY \cdot TP = TS \cdot TN.$$

10.  $SE$  is half  $TP$ ,

and  $PT^2 = PN^2 + TN^2 = 4AS \cdot AN + 4AN^2$ ;

$$\therefore SE^2 = AN \cdot XN = AN \cdot SP.$$

11. If  $SY$  be drawn perpendicular to the tangent and  $A$  be vertex,  $SA Y$  is a right angle ;

$\therefore A$  lies on the circle of which  $SY$  is diameter.

12. Draw  $SY$  perpendicular to the tangent, then if the circle described with centre  $S$  and radius equal to a quarter of the latus rectum meet the circle described on  $SY$  as diameter in  $A$ ,  $A$  is the vertex.

Produce  $SA$  to  $X$ , making  $AX$  equal to  $SA$ , then  $X$  is the foot of the directrix.

$$13. \quad SN : SP :: SN' : SP',$$

$$\text{or } AN - AS : AN + AS :: AS - AN' : AS + AN';$$

$$\therefore AN : AS :: AS : AN';$$

$$\therefore AN \cdot AN' = AS^2.$$

Again,  $4AS \cdot AN : 4AS^2 :: 4AS^2 : 4AS \cdot AN'$ ;

$$\therefore PN^2 : SR^2 :: SR^2 : P'N'^2,$$

or  $PN : SR :: SR : P'N'$ ;

$\therefore$  the latus rectum is a mean proportional between the double ordinates.

14. Let  $V$  be the middle point of the focal chord  $PSP'$ , and let the diameter through  $V$  meet the curve in  $Q$ ; then, if  $QT$ ,  $QM$  be the tangent and ordinate at  $Q$ , and  $VL$  be ordinate of  $V$ ,

$$VL = QM \text{ and } TM = SL;$$

$$\therefore VL^2 = QM^2 = 4AS \cdot AM = 2AS \cdot TM = 2AS \cdot SL.$$

Hence the locus of  $V$  is a parabola of which  $S$  is vertex and  $SL$  axis.

15. If  $P, P'$  be the given points,  $PK, P'K'$  perpendiculars on the directrix, the focus is the point of intersection of a circle centre  $P$ , radius  $PK$  with a circle of which  $P'$  is centre and  $P'K'$  radius.

In general two circles intersect in two points, therefore two parabolas can be drawn satisfying the given conditions.

16. If  $PG$  be normal at  $P$ , the triangles  $PNG, pPR$  are similar;

$$\therefore Pp : PN :: RP : NG;$$

$$\therefore RP = 2NG = \text{latus rectum};$$

$\therefore$  the locus of  $R$  is an equal parabola having its vertex  $A'$  on the opposite side of  $X$ , such that  $AA'$  is equal to the latus rectum.

17. Let  $P, P'$  be the given points,  $S$  the given focus.

A common tangent to the circles described with centres  $P, P'$  and radii  $PS, P'S$  respectively will be the directrix.

18. If  $SP$  be the focal distance and  $SY$  perpendicular to the tangent at  $P$ ,  $Y$  lies on the circle of which  $SP$  is diameter.

Also the angle  $AYS =$  the angle  $SPY$ ;

$\therefore AY$  touches the circle.

19. The tangents at the ends of the focal chord  $PSP'$  meet in  $F$  on the directrix at right angles : also the straight line through  $F$  at right angles to the directrix bisects  $PP'$  in  $V$ ;

$$\therefore FV = VP = VP' ;$$

$\therefore$  the directrix touches the circle described on  $PP'$  as diameter.

20. Draw the farther tangent to the circle parallel to the given diameter, then the locus of the point is a parabola of which the centre is focus, and the tangent thus drawn directrix.

21. Draw a straight line parallel to the given straight line, on the farther side of it, and at a distance from it equal to the radius of the circle, then the locus of the point is a parabola of which the centre of the circle is focus, and the straight line thus drawn directrix.

22. Let  $Q$  be the centre of the circle touching the sector in  $R$  and  $AC$  in  $M$ .

Through  $C$  draw  $CB$  at right angles to  $AC$ , and on the same side of it as  $Q$ , and draw  $QN$  perpendicular to the tangent at  $B$ .

Then

$$NQ + QM = BC = CQ + QR ;$$

$$\therefore CQ = QN ;$$

$\therefore Q$  lies on a parabola of which  $C$  is focus and  $BN$  directrix.

23.  $Y$  is the middle point of  $TP$  and  $Z$  of  $PG$  ;  
therefore  $YZ$  is parallel to the axis.

24. If  $SQ$  be perpendicular to the normal  $PG$ ,

$$PQ = QG,$$

and if  $QM$  be the ordinate,  $NM = MG$  ;

$$\therefore SM = AN \text{ and } PN = 2QM ;$$

$$\therefore QM^2 = AS \cdot AN = AS \cdot SM ;$$

$\therefore Q$  lies on a parabola of which  $S$  is vertex and  $SG$  axis.



25. The triangle  $PSG$  is isosceles; therefore  $GL$  is equal to  $PN$ .

26. If the circle described with centre  $S$  and radius equal to the perpendicular from  $S$  on the tangent at  $P$  meet the circle of which  $SP$  is diameter in  $Y$ , and the angle  $SYA$  be made equal to the angle  $SPY$ , then the foot of the perpendicular  $SA$  on  $YA$  will be the vertex.

27. Since  $SQ$  is double  $SA$ ,  $ASQ$  (and likewise  $QSP$ ) is equal to the angle of an equilateral triangle; therefore  $SP$  and  $SQ$  are equally inclined to the latus rectum.

$$\begin{aligned}
 28. \quad QX^2 &= SX^2 + SQ^2 + 2SX \cdot SQ \\
 &= 4AS^2 + QG^2 + 2SQ \cdot NG \\
 &= 4AS^2 + QN^2 + 2QN \cdot NG + NG^2 + 2SQ \cdot NG \\
 &= 4AS^2 + QN^2 + NG^2 + 2NG \cdot SN \\
 &= 4AS^2 + NQ^2 + 2AN \cdot NG \\
 &= 4AS^2 + QN^2 + PN^2 = 4AS^2 + QP^2.
 \end{aligned}$$

29. The angle

$$SPF = SPG - FPG = SGP - GPH = SHP;$$

therefore the triangles  $SPF$ ,  $SHP$  are similar;

$$\therefore SF \cdot SH = SP^2 = SG^2.$$

30.  $A, B, C, S$  lie on a circle; therefore, if  $D$  be the end of the diameter drawn through  $S$ ,  $DA, DB, DC$  are perpendicular to  $SA, SB, SC$  respectively.

31. Since  $PQ$  and  $PR$  are equally inclined to the axis, the circle through  $P, Q, R$  touches the parabola at  $P$ ; therefore  $PQ$  is a diameter of this circle.

Therefore  $PRQ$ , the angle in a semicircle, is a right angle.

32. Let  $MR$  and  $AQ$  meet in  $V$ .

Draw the ordinates  $VW, RZ$ .



Then  $MW : MZ :: WV : RZ :: AW : AN$ ;  
 $\therefore MW : AW :: MZ : AN$ ;  
 $\therefore AN : AW :: AZ : AN$ .

Again,  $VW^2 : QN^2 :: AW^2 : AN^2$ ;  
 $\therefore VW^2 : RZ^2 :: AW : AZ$ ;  
 $\therefore V$  lies on the curve.

33. Let  $P, Q$  be the given points. Bisect  $PQ$  in  $V$ , and draw  $VT$  parallel to the axis meeting the given tangent  $P$  in  $T$ .

Draw  $PS, QS$  such that  $TP, TQ$  may be equally inclined to the axis and to  $SP, SQ$  respectively.  $PS, QS$  meet in the focus.

Through  $P$  draw a straight line  $PK$  parallel to the axis, making  $PK$  equal to  $SP$ , then the straight line through  $K$  at right angles to  $PK$  will be the directrix.

34. Let  $P$  be the vertex and  $QVQ'$  be corresponding ordinate.

Take  $M$  in  $VP$  produced such that

$$QV^2 = 4MP \cdot PV.$$

Make angle  $TPS$  equal to the angle  $MPT$ ,  $PT$  being parallel to  $QQ'$ , and make  $PS$  equal to  $PM$ .

Then  $S$  is the focus, and the straight line drawn through  $M$  at right angles to  $PM$  is the directrix.

35.  $PM^2 : QN^2 :: AM : AN$ ,  $QN$  being the ordinate of  $Q$ ;

$$\therefore AM = 4AN \text{ and } 3AM = 4NM;$$

$$\therefore 3AT = 4QN = 2PM.$$

36. Draw  $PN$  perpendicular to  $AB$ .

Then  $AN : NP :: CQ : AC :: NP : AC$ ;

$$\therefore PN^2 = AC \cdot AN.$$

Therefore the locus of  $P$  is a parabola of which  $A$  is vertex and  $AB$  axis.

37. The triangles  $LKP$ ,  $PSK$ ,  $KSA$  and  $TKA$  are similar ;

$$\therefore KL^2 : SP^2 :: KP^2 : KS^2 :: KA^2 : AS^2 :: TA : AS \\ :: SP - AS : AS.$$

38. With centre  $S$  and radius one-fourth of the chord describe a circle meeting the parabola in  $P$ . The chord through  $S$  parallel to the tangent at  $P$  will be the chord required.

$$39. PN^2 = 4AS. AN = 4AS. AN' + 4AS^2 \\ = P'N'^2 + N'G'^2 = P'G'^2.$$

40. If  $Pp$ ,  $P'p'$  be two parallel chords, and  $V$ ,  $V'$  their middle points,  $VV'$  is a diameter. Let  $VV'$  meet the curve in  $Q$ .

Draw  $QT$  parallel to  $Pp$ , then  $QT$  is the tangent at  $Q$ .

Produce  $VV'$  to a point  $M$  such that  $PV^2 = 4QM$ .  $QV$ , then the straight line drawn through  $M$  at right angles to  $MV$  is the directrix.

Make the angle  $TQS$  equal to the angle  $TQM$ .

Then if  $QS$  be made equal to  $QM$ ,  $S$  is the focus and the straight line through  $S$  perpendicular to the directrix is the axis.

41. Let the tangents at  $P$  and  $P'$  intersect in  $T$ .

$$\text{Then} \quad 4SP.PV : 4SP'.P'V' \\ :: P'V^2 : PV'^2 :: TP^2 : TP'^2 :: SP : SP' ; \\ \therefore PV = P'V'.$$

42. If in the preceding Example  $P'T$  meets  $PV$  in  $Z$  and the sides of the triangle  $ABC$  are parallel to  $ZP$ ,  $PT$  and  $TZ$  respectively,

$$AB : AC :: TZ : TP :: TP' : TP.$$

43. If  $U$ ,  $V$  be the vertices of the diameters bisecting  $Pp$ ,  $Qq$ ,

$$PS.Sp : QS.Sq :: SU : SV :: Pp : Qq.$$

44. Draw  $RW, LZ$  parallel to  $QQ'$ .

Then 
$$PL^2 : PR^2 :: LZ^2 : RW^2$$
$$:: QV^2 : RW^2 :: PV : PW :: PN : PR ;$$
$$\therefore PL^2 = PR \cdot PN.$$

45. This question is solved in Conics, Art. 212, p. 217.

46. 
$$PN^2 : AN^2 :: AM^2 : QM^2 ;$$
$$\therefore 4AS : AN :: AM : 4AS.$$

47. Let  $AP, Ap$  meet the latus rectum in  $L$  and  $l$  respectively.

Then 
$$PN^2 : SL^2 :: AN^2 : AS^2 :: AN : An, \text{ (Example 13)}$$
$$:: PN^2 : pn^2 ;$$
$$\therefore SL = pn.$$

In like manner 
$$Sl = PN.$$

48. If  $PK, QL$  be perpendicular to the directrix, and  $QL'$  to  $PK$  produced, the angle  $SPQ =$  the angle  $QPL'$  ;

$$\therefore PL' = SP = PK ;$$
$$\therefore SQ = QL = KL' = 2PK = 2SP.$$

49. Is equivalent to Example 32.

50. Let  $Q$  be the point of intersection, and let  $QK$  be the ordinate of  $Q$ .

Then 
$$AK : QK :: PN : NT ;$$
$$\therefore 2AK \cdot AN = QK \cdot PN = PN^2 = 4AS \cdot AN ;$$
$$\therefore AK = 2AS,$$

or  $Q$  lies on a fixed straight line parallel to the directrix.

51. Let  $Tp, Tq$  be the fixed tangents, and let  $PQ$  touch the curve in  $R$ .

Then 
$$SP^2 = Sp \cdot SR = Sq \cdot SR = SQ^2 ;$$
$$\therefore SP = SQ.$$

52. Let  $TM$  be the ordinate of  $T$ , and  $TW$  perpendicular to  $SP$ .

$$\begin{aligned} TM^2 &= ST^2 - SM^2 = TW^2 + SW^2 - SM^2 \\ &= TW^2 + XM^2 - SM^2 = TW^2 + XS^2 + 2XS \cdot SM; \end{aligned}$$

$\therefore$  the locus of  $T$  is a parabola of which  $XS$  is axis.

If  $TW = 2AS$ ,  $TM^2 = 4AS \cdot XM$ , or  $X$  is the vertex.

53. Let the chord  $PQ$  meet the axis in  $O$ , and the tangent at  $A$  in  $V$ .

Then by Art. 48,  $VO^2 = VP \cdot VQ$ ;

$\therefore V$  is a fixed point, and the locus of  $A$  is the circle of which  $OV$  is diameter.

54. Let the diameter  $TV$  meet the curve in  $R$ .

Then the tangent at  $R$ , being parallel to  $PQ$ , meets  $TP$  at right angles in  $Z$  on the directrix.

Also  $TZ : ZP :: TR : RV$ ;

$$\therefore TZ = ZP.$$

Therefore  $T$  and  $P$  are equidistant from the directrix.

55. Let  $PT$  meet the axis in  $t$ .

Then  $PQ : PT :: 2PV : PT :: 2PG : Pt,$   
 $:: 2PN : Nt :: PN : AN.$

56. If the tangents  $TP$ ,  $TQ$  are equal,  $T$  lies on the axis.

Let the tangent at  $R$  meet them in  $p$  and  $q$ .

Then, since  $T$ ,  $p$ ,  $q$  and  $S$  lie on a circle, the triangles  $SqT$ ,  $SpP$  are similar;

$$\therefore Tq : pP :: TS : SP;$$

$$\therefore Tq = pP.$$

So  $Tp = qQ.$

$$57. \quad AN \cdot NL = PN^2 = 4AS \cdot AN;$$

$$\therefore NL = 4AS.$$

But  $NG = 2AS;$

$$\therefore LG = \text{half the latus rectum.}$$

58. By Art. 5,  $P'S$ ,  $Q'S$  are the external bisectors of the angles  $PSA$ ,  $QSA$ ;

therefore  $P'SQ'$  is a right angle.

59. The angles  $TCS$ ,  $DRS$  are equal, being supplements of equal angles  $SCP$ ,  $SRC$ , Art. 35.

$$\text{And the angle } CTS = TQS = RDS;$$

$$\therefore \text{the triangles } TCS, DRS \text{ are similar;}$$

$$\therefore DR : TC :: RS : SC :: RC : CP;$$

$$\therefore PC : CT :: CR : RD.$$

Similarly  $TD : DQ :: CR : RD.$

60. Let  $AD$  and  $XP$  intersect in  $Q$ , and let  $QM$  be the ordinate.

$$\text{Then } QM : DS :: AM : AS \text{ and } QM : PN :: XM : XN;$$

$$\therefore AM : XM :: AS : XN;$$

$$\therefore AM : AS :: AS : AN;$$

$$\therefore QM^2 : PN^2 :: AM^2 : AS^2 :: AM : AN,$$

or  $Q$  is on the parabola.

61. By Example 18,  $YY'$ , the tangent at the vertex, is a common tangent.

$$SY^2 = AS \cdot SP, \quad SY'^2 = AS \cdot Sp;$$

$$\therefore YY'^2 = SY^2 + SY'^2 = AS \cdot Pp.$$

62. If  $PV$  be the diameter bisecting  $AQ$ ,

$$AM = 4AN.$$

Also  $AM \cdot MR = QM^2 = 4AS \cdot AM;$   
 $\therefore MR = 4AS.$

Now focal chord parallel to  $AQ$   
 $= 4SP = 4XN = 4AS + AM = AR.$

63. Let  $AR, CP$  meet in  $p$ .

Draw  $pN, pD$  perpendicular to  $CA, CR$ , and let  $Dp$  meet the tangent at  $A$  in  $M$ .

$$Cp : CP :: CD : CR :: Np : CR :: AN : AC;$$

$$\therefore Cp = AN = pM.$$

Therefore the locus of  $p$  is a parabola of which  $C$  is focus and  $AM$  directrix.

64. If  $QMQ'$  be the common chord,

$$9AS'^2 = 4AQ^2 = 4AM^2 + 4QM^2 = 4AM^2 + 16AM \cdot AS;$$

$$\therefore AM \text{ is half } AS.$$

65. Let the fixed straight line  $BK$  meet the tangent at  $P$  in  $K$ .

Draw  $KY'$  at right angles, and  $SY'$  parallel to  $KP$ .

Draw  $Y'A'$  perpendicular to the axis, and  $KL$  parallel to  $BA$ .

Then, since  $KY = SY', SA' = KL = BA$

therefore  $A'$  is a fixed point.

Therefore  $KY'$  touches the parabola of which  $S$  is focus and  $A'$  vertex.

$$66. QD \cdot DR = QM^2 - DM^2 = QM^2 - PN^2$$

$$= 4AS \cdot AM - 4AS \cdot AN = 4AS \cdot PD.$$

67. Draw the double ordinate  $QMq$ ; then, if the diameter through  $Q'$  meet  $Qq$  in  $D'$ ,

$$QD' \cdot D'q = 4AS \cdot Q'D'.$$



Now  $NT : PN :: Q'D' : QD' :: D'q : 4AS;$

$\therefore D'q : 4AS :: 2AN : PN :: PN : 2AS;$

$\therefore 2PN = D'q = D'M + Mq = QM + Q'M'.$

Therefore the line through  $P$  bisecting  $QQ'$  is parallel to the axis.

Hence the locus of the middle points of a series of parallel chords is a straight line parallel to the axis.

68<sup>1</sup>. Take  $CP, CQ$  two tangents such that  $PCQ$  is two-thirds of a right angle; join  $SC$  cutting the curve in  $R$ , and draw the tangent  $ARB$ . Then, Art. 38,

$CSP = CSQ = 120^\circ$ , and  $CAR = \frac{1}{2}CSQ = 60^\circ;$

$\therefore CAB$  is equilateral.

69. Draw  $AZ, AN$  perpendicular to the tangent and  $SY$  respectively, and draw  $SM$  perpendicular to  $ZA$ .

Then  $SM^2 = AN^2 = YN \cdot NS = ZA \cdot AM.$

Therefore the locus of  $S$  is a parabola of which  $A$  is vertex and  $ZM$  axis.

70. If  $GZ$  be drawn parallel to  $PY$  and  $SZ$  to  $PG$ , then  $SY, SZ$  are equal.

Therefore, if  $ZB$  be perpendicular to the axis,  $BS = AS$ .

Hence  $GZ$  touches an equal parabola of which  $B$  is vertex and  $S$  focus.

71. If  $pqr$  be the triangle formed by the given straight lines, describe a parabola passing through  $p, q$  and  $r$  having its axis parallel to  $AS$ . (Ex. 45.)

If  $s$  be the focus of this parabola, draw  $SP$  parallel to  $sp$ ,  $PQ$  to  $pq$ , and  $PR$  to  $pr$ .

Then  $PQ : pq :: SA : sa :: PR : pr,$

and the angles  $QPR, qpr$  are equal.

<sup>1</sup> If a parabola touch the sides of an equilateral triangle, the focal distance of any vertex of the triangle passes through the point of contact of the opposite side.



72. Let  $RW$  be the ordinate of  $R$ .

Then

$$AN^2 : AW^2 :: PN^2 : RW^2 :: PN^2 : QM^2 :: AN : AM ;$$

$$\therefore AN : AW :: AW : AM,$$

or

$$WN : AN :: MW : AW ;$$

$$\therefore RL : QR :: AN : AW :: PN : NL.$$

73. Let  $PV$  be the ordinate to the diameter  $RQM$ .

Then

$$PM : RM :: PN : TN$$

$$:: 2PN \cdot AS : 4AS \cdot AN :: 2AS : PN ;$$

$$\therefore PM \cdot PN = 2AS \cdot RM.$$

But

$$PM^2 = 4AS \cdot QV = 4AS \cdot RQ ;$$

$$\therefore RM : RQ :: 2PN : PM :: PP' : PM ;$$

$$\therefore QM : QR :: P'M : PM.$$

74. Let  $PP'$  be the chord,  $TWV$  its diameter,  $RQM$  the line parallel to the axis.

Then  $PM : RM :: PV : TV :: PV : 2WV$

$$:: 2VP \cdot SW : 4SW \cdot WV :: 2SW : PV ;$$

$$\therefore PM \cdot PV = 2SW \cdot RM.$$

But

$$PM \cdot MP' = 4SW \cdot QM,$$

$$\therefore RM : QM :: 2PV : MP',$$

or

$$RQ : QM :: PM : MP'.$$

75.  $SR, Sr$  are the exterior bisectors of the angles  $PSQ, pSQ$  respectively.

Therefore  $RSr$  is a right angle.

Therefore  $SD$ , which is half the latus rectum, is a mean proportional between  $DR$  and  $Dr$ .

76. Let  $PVP'$  be parallel to the given straight line,  $QV'Q'$  the chord joining the two other points of intersection of the parabola and circle.

Let the diameters through  $V$  and  $V'$  meet the curve in  $p$  and  $p'$ .

Then  $pp'$  is a double ordinate; draw  $V'H$  parallel to  $pp'$  to meet  $pV$ .

$VV'$  is perpendicular to  $QQ'$ , and therefore parallel to the normal at  $p'$ ;

$$\therefore VV' : p'g :: V'H : p'n ;$$

$$\therefore VV' = 2p'g.$$

77. The arcs  $QU$  and  $RV$  are equal, since  $QV$  and  $UR$  are parallel.

Therefore  $QR$  and  $UV$  are equally inclined to  $QV$ , that is to the axis.

But  $QR$  and the tangent at  $P$  are equally inclined to the axis ;

therefore  $UV$  is parallel to the tangent at  $P$ .

$$\begin{aligned} 78. \quad VR : VR' :: VR : V'Q' :: PV : PV' \\ \quad \quad \quad :: QV^2 : Q'V'^2 :: QV^2 : VR'^2 ; \\ \therefore VR \cdot VR' = QV^2. \end{aligned}$$

79. If  $FR$ ,  $QE$  meet the tangent at  $P$  in  $V$  and  $T$ ,

$$TE : EQ :: VR : RF :: PF : FQ. \quad (\text{Ex. 74.})$$

Therefore  $EF$  is parallel to  $TP$ .

80. If  $Q$  be the vertex of the diameter bisecting the chord  $Rr$  which meets the diameter  $PW$  in  $W$ ,

$$RW \cdot Wr = 4SQ \cdot PW.$$

Therefore the rectangle under the segments varies as the distance of the point of intersection  $W$  from  $P$ .

81.  $QS$ ,  $Q'S$  are equally inclined to  $SP$ , and therefore to the axis.

Therefore  $Q'S$  meets the curve at the end of the double ordinate  $QMq$ , and, since  $AM \cdot AM = AS^2$ , the semi-latus rectum is a mean proportional between  $QM$  and  $Q'M'$ .

Also, since the diameter through  $P$  bisects  $QQ'$ ,  $PS$  is an arithmetic mean between  $QM$  and  $Q'M'$ .

82.  $BB'$  will bisect  $C'A'$  in  $V$ .

Let  $V'$  be the middle point of  $B'B''$ .

$VV'$  is parallel to the axis.

And  $BB''$  is parallel to  $VV'$ , and therefore to the axis.

Similarly  $AA''$  and  $CC''$  are parallel to the axis.

83. Let  $C$  be the centre of the circle.

The angle between tangents to circle  $= PCP' = 2PSP'$   
 $= 4$  times angle between the tangents to the parabola.

84. The tangents at the ends of the focal chord  $PSP'$  will meet in  $T$  on the directrix.

If the normals at  $P$  and  $P'$  meet in  $Q$ ,  $TQ$  will be parallel to the axis.

Let  $TQ$  meet the curve in  $p$  and  $PP'$  in  $V$ . Let  $QM$  be the ordinate of  $Q$ .

Then  $XM = TQ = 2TV = 4Sp = 4Xn$ .

Therefore, if we take  $B$  in  $XM$  such that  $XB = 4AS$ ,

$$BM = 4An, \quad QM^2 = pn^2 = 4AS \cdot An = AS \cdot BM.$$

Hence the locus of  $Q$  is a parabola of which  $B$  is vertex and  $BM$  axis.

85. Produce  $PA$  to  $P'$ , making  $AP'$  equal to  $AP$ .

On  $AP'$  as diameter describe a circle meeting the tangent at  $P$  in  $T$ .

Join  $TA$  and produce to  $N$ , making  $AN$  equal to  $AT$ .

In  $AN$  take a point  $S$  such that  $PN^2 = 4AS \cdot AN$ , then  $S$  is focus.

86. If  $G$  be the intersection of the normals and  $Q$  vertex of the diameter bisecting  $PSp$ ,

$$PS \cdot Sp = AS \cdot Pp = AS \cdot TG.$$

87. If  $pq$  be a tangent parallel to  $PQ$ ,  $Tp = pP$ , and  $T, p, q, S$  and  $O$  lie on a circle.

Therefore the angles  $TSO, TpO$  are equal, and  $TpO$  is a right angle.

$$88. \quad SM^2 : AN^2 :: QM^2 : PN^2 :: AM : AN ; \\ \therefore SM^2 = AM \cdot AN.$$

So  $SM'^2 = AM' \cdot AN ;$   
 $\therefore MM' \cdot AN = MM' \cdot (SM - SM') ;$   
 $\therefore SM - SM' = AN.$   
 $MM' = SQ - SQ' ;$   
 $\therefore MM' : SM - SM' :: AP : AN ;$   
 $\therefore MM' = AP.$

89. If  $P, Q, P', Q'$  be the points of intersection,  $PQ, P'Q'$  are equally inclined to the axis.

Hence the middle points of  $PQ$  and  $P'Q'$  are equidistant from the axis.

Therefore, if  $P, Q$  be on one side of the axis and  $P'Q'$  on the other, the sum of the ordinates of  $P$  and  $Q$  is equal to the sum of the ordinates of  $P'$  and  $Q'$ .

If  $P'$  be on the same side of the axis as  $P$  and  $Q$ , the ordinate of  $Q'$  is equal to the sum of the ordinates of  $P, Q$ , and  $P'$ .

90. Let the diameter through  $T$  meet the curve in  $W, PQ$  in  $V$ , and  $PN$  in  $t$ .

Let  $WZ$  be the ordinate of  $W$ ; draw  $Qq$  parallel to the axis to meet  $PN$ .

$$QM \cdot PN = PN \cdot qN = tN^2 - Pt^2 = WZ^2 - 4AS \cdot WV \\ = 4AS \cdot AZ - 4AS \cdot LZ = 4AS \cdot AL.$$

$$91. \quad pX : XA :: PN : AN :: 4AS : PN \\ :: 4AS \cdot QM : 4AS \cdot AL. \quad (\text{Ex. 90.})$$

So  $qX : XA :: PN : AL ;$   
 $\therefore pX + qX : XA :: PN + QM : AL ;$   
 $\therefore pX + qX : PN + QM :: XA : AL :: tX : TL.$

But  $NP + QM = 2TL ;$   
 $\therefore pX + qX = 2tX,$   
 or  $pt = tq.$

92. Let  $TF$ ,  $TD$  be drawn parallel to  $PE$ ,  $QE$  normals at  $P$  and  $Q$ .

The angle  $TFQ = PEQ =$  supplement of  $PTQ = TSQ$ ;

$\therefore Q, S, F, T$  lie on a circle.

Therefore  $TSF$  is a right angle.

So  $TSD$  is a right angle, and  $DF$  goes through  $S$ .

93. If  $pq$  be a tangent parallel to  $PQ$ ,  $Tq = qQ$ .

Also,  $T, p, q$  and  $S$  lie on a circle;

therefore the angles  $Tpq, TSq$  are equal.

Therefore  $TSq$  is a right angle.

94. Let  $RO$  be the diameter through the given point  $O$ .

Take  $T$  in  $OR$  produced such that  $TR : RO$  in the given ratio.

If  $TP$  be a tangent, the chord  $POQ$  will be divided as required. (Ex. 74.)

95. If  $QN$  be the ordinate,

$$BP + PQ = QN + BX - NX = BX + QN - SQ,$$

which is greatest when  $QN = SQ$ , that is when  $Q$  is on the latus rectum.

96. If  $SZ$  and  $PG$  meet in  $Q$  and  $QT$  be ordinate,

$$TA : AS :: QZ : ZS :: QP : PG :: TN : NG;$$

$$\therefore TN = 2TA.$$

97. If  $QV$  be the ordinate of the point of contact,

$$TP = PV.$$

Therefore the distance of  $V$  from  $TQ$  is twice the distance of  $P$ , or the locus of  $V$  is a straight line parallel to  $TQ$ .

98. If  $TPSQ$  be the parallelogram, the angles  $TSP, TSQ$  are equal;

therefore  $TPSQ$  is a rhombus and  $T$  lies on the axis.

Therefore  $TSP$  is the angle of an equilateral triangle.

99. If  $SZ$ ,  $SZ'$  be the perpendiculars on the second tangents,  $TQ$ ,  $TQ'$  and  $PP'$  be the common tangent,  $SY$  perpendicular to it,

then angle  $A'SY = ASY = YSP$  ;  
 $\therefore A'$  lies in  $SP$ , and  $A$  in  $SP'$  ;  
 $\therefore SP = SP'$ .

Now  $SQ \cdot SP = ST^2 = SQ' \cdot SP'$  ;  
 $\therefore SQ = SQ'$  ;  
 $\therefore SZ = SZ'$ .

100. If the tangent meet  $AY$  in  $Y$  and the other parabola in  $Q$ ,

$$QM^2 = \frac{1}{2} AS \cdot AM, \quad AY^2 = AS \cdot AT,$$

$$QM : AY = MT : AT ;$$

$$\therefore 2TM^2 = AT \cdot AM.$$

This can be constructed by taking  $AM = MT$ , or by taking  $AM = 2AT$ , the two solutions corresponding to the two points in which the parabola is cut by the tangent.



## CHAPTER III.

### THE ELLIPSE.

1.  $SD^2 = BC^2 = CS.SX.$

Therefore  $CDX$  is a right angle.

2.  $ST, SP$  are equally inclined to  $PT$ , since  $pST$  is parallel to  $S'P$ .

Therefore  $ST = SP.$

3.  $PN : PG' :: SY : SP :: BC : CD$   
 $:: PF : AC :: AC : PG'.$

Therefore  $PN = AC.$

4.  $T$  lies on a circle of which  $QQ'$  is a diameter and  $V$  centre ;

therefore  $VT = VQ.$

Now  $QV^2 : CP^2 - CV^2 :: CD^2 : CP^2,$

or  $VT^2 : CV.CT - CV^2 :: CD^2 : CP^2.$

Therefore  $TV : VC :: CD^2 : CP^2.$

Therefore  $CT : CV :: CD^2 + CP^2 : CP^2,$

or  $CT^2 : CV.CT :: AC^2 + BC^2 : CP^2.$

Therefore  $CT^2 = AC^2 + BC^2.$

5. Through  $T$  draw a straight line at right angles to  $AA'$  meeting  $AP, A'P$  in  $E, E'.$

Then  $ET : PN :: AT : AN :: CT - CA : CA - CN.$



Now  $CT : CA :: CA : AN$ ;  
 $\therefore CT + CA : CT - CA :: CA + AN : CA - CN$ ;  
 $\therefore ET : PN :: A'T : A'N :: ET : PN$ .

Hence  $PT$  bisects any straight line parallel to  $ET$  terminated by  $A'P$ ,  $AP$ .

6. Draw  $CD$  parallel to the given line, and  $CP$  parallel to the tangent at  $D$ .

The tangent at  $P$  will be parallel to  $CD$  and the given line.

7.  $SR : XE :: SA : AX :: SR : SX$ .  
 Therefore  $XE = SX$ ,  
 and  $AT = AS$ .

8. Draw  $GL$  perpendicular to  $SP$ .  
 Then  $PL = SR$ ,  
 and  $SY : SP :: PL : PG :: SR : PG$ .

The angle  $SPS'$  is greatest when  $SPY$  is least, that is when  $SY : SP$  or  $BC : CD$  is least.

Hence  $SPS'$  is greatest when  $CD$  is greatest, that is when  $CD = AC$ .

Hence  $SPS'$  is greatest when  $P$  is on the minor axis.

9.  $CE'^2 = CP^2 + PE'^2 + 2PF \cdot PE' = CD^2 + CP^2$   
 $+ 2CD \cdot PF = AC^2 + BC^2 + 2AC \cdot BC$ ;  
 $\therefore CE' = AC + BC$ .

So  $CE = AC - BC$ .  
 $(CP + CD)^2 = AC^2 + CB^2 + 2CP \cdot CD$ ,

which is greater than  $(AC + BC)^2$ , since  $CP \cdot CD$  is greater than  $PF \cdot CD$  or  $AC \cdot BC$ .

Similarly  $CP - CD$  is less than  $AC - BC$ .

10. Let  $S'Q$  drawn parallel to  $SP$  meet the normal in  $K$ , and  $SY$  in  $Q$ .

Then  $S'K = S'P$  and  $KQ = SP$ ;

therefore  $S'Q = AA'$ .

$$11. \quad SY : S'Y' :: YP : PY' :: TP - TY : TY' - TP \\ :: PG - SY : S'Y' - PG.$$

12.  $PS'Q$  is the supplement of  $QPS' + PQS'$ ,  
and is therefore equal to the excess of twice  $QPT + PQT$   
over two right angles,

that is, is the supplement of twice  $PTQ$ .

13. Since  $CZ$  and  $SP$  are parallel, the angle

$$CZP = SPY = SNY;$$

therefore  $Y, Z, C, N$  lie on a circle.

14. Let  $AQ$  and  $SP$  meet in  $R$ .

Then  $SA : SR :: SG : SP :: SA : AX$ .

Therefore  $R$  lies on a circle of which  $S$  is centre.

15. Since  $KPt$  is a right angle,  $t$  lies on a circle which  
passes through  $S, P, S', K$ ,

therefore  $GK : SK :: S'G : SP :: SA : AX$ ,

and  $St : tK :: SY : SP :: BC : CD$ .

16. If  $SP$  meet  $S'Y'$  in  $Z$ , then since  $S'Y' = Y'Z$ ,  
 $SY'$  will bisect  $PG$ .

17. Let the circle whose centre is  $P$  touch the circles  
whose centres are  $S, H$  in  $Q, R$ .

Then  $SP + PH = SQ + QP + PH = SQ + HR$ .

Hence the locus of  $P$  is an ellipse of which  $S$  and  $H$  are  
foci.

$$18. \quad TN : TC :: PN : Ct.$$

Therefore  $TN \cdot NG : CT \cdot NG :: PN^2 : Ct \cdot PN$ .

But  $PN^2 = TN \cdot NG$ .

Therefore  $CT \cdot NG = Ct \cdot PN = CB^2$ .

$$19. \quad TP : TQ :: CD : AC :: BC : PF :: PG : BC.$$

$$\begin{aligned} 20. \quad PN^2 : AF \cdot A'F' &:: TN^2 : TA \cdot TA' \\ &:: TN^2 : CT^2 - CA^2 :: TN : CT \\ &:: CT - CN : CT :: CA^2 - CN^2 : CA^2; \\ &\therefore AF \cdot A'F' = BC^2. \end{aligned}$$

21. The perpendiculars from  $T$  on  $SP, SQ, HP, HQ$  are all equal.

Hence a circle can be described with centre  $T$  to touch  $SP, SQ, HP, HQ$ .

22. If  $P, Q$  be two points of intersection,  $PC$  bisects the angle  $ACa$  and  $QC$  bisects  $A'Ca$ .

Therefore  $PCQ$  is a right angle.

23. If  $SP, HQ$  meet in  $R$ ,

$$PSQ + PHQ = 2PRQ - SQH - SPH,$$

and  $SQH + SPH + 2RQT + 2RPT = 4$  right angles,

$$\therefore PSQ + PHQ = \text{twice the supplement of } QTP.$$

24. Since  $t, P, S, g$  lie on a circle, the angle  $PSt = Pgt = STP$ .

$$25. \quad Q'M : PM :: BC : AC :: PN : QN.$$

Therefore  $Q'M : CN :: CM : QN$ .

Therefore  $QQ'$  passes through  $C$ .

$$26. \quad SY : SP :: BC : CD.$$

Therefore  $SY \cdot CD = SP \cdot BC$ .

27. If  $T$  be intersection of tangents at  $A$  and  $B$ , then, since  $TC$  bisects  $AB$ , it is a diameter of the conic. Therefore the tangent at  $C$  is parallel to  $AB$ .

28. The angles  $SPT, HPt$  are equal.

Also  $TP \cdot Pt = CD^2 = SP \cdot PH$ ,

or  $TP : SP :: HP : Pt$ .

Therefore  $SPT, HPt$  are similar.

29.  $PE = PE' = AC.$

Therefore  $SE = HE'$ , and the angles  $SCE, HCE'$  are equal. Therefore the circles circumscribing  $SCE, HCE'$  are equal.

30. The angles  $KPG, GPL$  are equal ; therefore  $KL$  is a double ordinate of the circle of which  $PG$  is diameter.

31. If  $Q$  be the centre,  $QN$  the ordinate, and  $T, T'$  the points where the tangent at  $P$  meets the tangents at the vertices,

$$QN^2 : SN.NH :: AT.A'T' : AH.A'S :: BC^2 : A'S^2. \quad (\text{Ex. 20.})$$

32. Since the tangents are equally inclined to  $SP, S'P$  respectively, the bisector of the angle between them bisects  $SPS'$ , and therefore passes through the point where the axis minor meets the circle.

33. If  $PQRS$  be the quadrilateral,  $p, q, r, s$  points of contact,  $H$  the focus,

$$\begin{aligned} \text{the angle } pHP = PHs, \quad pHQ = QHq, \\ SHr = SHs, \quad rHR = RHq. \end{aligned}$$

Therefore  $PHQ + SHR = PHS + QHR = \text{two right angles.}$

$$\begin{aligned} 34. \quad SG : SC :: SP :: SY \text{ (see Ex. 15)} \\ :: CD : BC :: PV : VA. \end{aligned}$$

35. The normals at  $P$  and  $Q$  will meet on the minor axis in  $K$ .

Then angle between the tangents  $= PKQ = PSQ.$

36. The auxiliary circle lies entirely without the ellipse except at  $A$  and  $A'$  ; therefore  $AA'$  is the greatest diameter.

The circle described on  $BB'$  as diameter lies wholly within the ellipse ; therefore  $BB'$  is the least diameter.

37. Let any circle passing through  $N$  and  $T$  meet the auxiliary circle in  $Q$ .

Then  $CN \cdot CT = CA^2 = CQ^2$ .

Hence  $CQ$  touches the circle at  $Q$ , and the circle cuts the auxiliary circle orthogonally.

38. The angle  $PNY = PSY = PS'Y' = PNY'$ .

Therefore  $PY : PY' :: NY : NY'$ .

39.  $PQ^2 : TQ^2 :: SY \cdot S'Y' : TY \cdot TY'$ .

But  $TQ^2 = TY \cdot TY'$ .

Therefore  $PQ^2 = SY \cdot S'Y' = BC^2$ .

Therefore  $PQ = BC$ .

40. If  $QN$  and  $PM$  be the perpendiculars on the given lines passing through  $C$ ,  $R$  their point of intersection,

$$RN : QN :: CP : CQ;$$

therefore the locus of  $R$  is an ellipse of which the outer circle is the auxiliary circle.

41.  $SP : S'P :: SY : S'Y' :: SY^2 : BC^2$ ,

and  $S'Q : SQ :: S'Z' : SZ :: BC^2 : SZ^2$ .

Therefore  $SP \cdot S'Q : S'P \cdot SQ :: SY^2 : SZ^2$ .

42. Let  $Ca$ ,  $Cb$  be the conjugate diameters, and  $Pm$ ,  $Pn$  ordinates of  $P$ .

Then  $Cm \cdot CM = Ca^2$ ,

and  $Cn \cdot CN = Cb^2$ ;

$$\therefore CM \cdot Pm : Ca^2 :: Cb^2 : Pn \cdot CN;$$

$\therefore$  the triangle  $CPM$  varies inversely as the triangle  $CPN$ .

43.

$CAV : CPT :: CA^2 : CT^2 :: CN : CT :: CPN : CPT$ ;  
therefore the triangles  $CAV$ ,  $CPN$  are equal.

44. Let  $TPQ$ ,  $Tpq$  be the tangents intersecting the auxiliary circle in  $P$ ,  $Q$ ,  $p$ ,  $q$ .

Let  $E$ ,  $e$  be their middle points.

$$\begin{aligned} PE^2 + pe^2 &= ET^2 + Te^2 - TP \cdot TQ - Tp \cdot Tq \\ &= CT^2 - 2TP \cdot TQ = CT^2 + 2CA^2 - 2CT^2 = SC^2. \end{aligned}$$

45. Let  $QQ'$  be a diameter equally inclined to the axis with the conjugate to  $PP'$ .

Then the circles described through  $P$ ,  $P'$ ,  $Q$  and  $P$ ,  $P'$ ,  $Q'$  will touch the ellipse at  $Q$  and  $Q'$ .

Hence  $Q$ ,  $Q'$  are the points at which  $PP'$  subtends the greatest and least angles respectively.

46. Draw the tangent  $Qr$ .

Then, since the angles  $PSQ$ ,  $QSr$  are equal,  $Q$  always lies on the tangent at the end of the focal chord  $RSr$ .

47. The triangle  $YCY'$  will be the greatest possible when  $YCY'$  is a right angle :  $P$  will then lie on the circle of which  $SS'$  is diameter.

This intersects the ellipse in four points, provided  $SS'$  is greater than  $BB'$ .

48. The points where the lines joining the foci of the two ellipses meet the common auxiliary circle are points through which the common tangents pass.

49. The circle passing through the feet of the perpendiculars is the auxiliary circle of the ellipse.

50. Draw  $QN$  perpendicular to  $AB$ .

Then  $QN : NA :: BP : AP :: CA : AT$ ,

and  $QN : BN :: AT : AB$ .

Therefore  $QN^2 : AN \cdot NB :: CA : AB$ .

Therefore the locus of  $Q$  is an ellipse of which  $AB$  is major axis.



51.  $PG, GN, NP$  are at right angles to  $CD, DR, RC$  respectively.

Therefore the triangles  $CDR, PGN$  are similar.

Therefore  $PG : CD :: PN : CR :: BC : AC$ .

52. Let  $PS, QS$  meet the ellipse and circle again in  $p, q$ .

And let  $P'Cp'$  be the diameter parallel to  $SP$ .

Then, since  $pq$  is an ordinate,

$$SQ : SP :: Sq : Sp :: Qq : pP :: AA' : P'p'.$$

Again,  $PS.Sp : AS.SA' :: CP'^2 : CA^2$ .

Therefore  $SR.Pp : 2BC^2 :: P'p'^2 : AA'^2$ ,

or  $Pp : AA' :: P'p'^2 : AA'^2$ .

Therefore  $SQ : SP :: AA' : Qq$ ,

and  $Qq = P'p'$ .

53. If  $SP$  meet  $S'Y'$  in  $L'$ ,  $SL' = AA'$ ;

therefore  $SR = AC$ .

54. Since the directions before and after impact are equally inclined to the tangent at the point of impact, the lines in which the ball moves will touch a confocal ellipse or hyperbola.

55. Let the tangent at  $P$  meet the tangents at  $A$  and  $A'$  in  $F$  and  $F'$ .

Then, since the angles  $PSF, FSA$  and  $PSF', F'SA'$  are respectively equal,  $S$  (and similarly  $S'$ ) lies on the circle of which  $FF'$  is diameter.

56.  $P', D'$ , the two angular points, will lie in  $PN, DM$  respectively.

Therefore  $P'N : NC :: DM : NC :: BC : AC$ .

Therefore  $P'$  lies on a fixed straight line through  $C$ .

Similarly  $Q'$  lies on the other equi-conjugate diameter.

57. The angles  $SPS', STS'$  are equal by Ex. 15.

$\therefore SPS' : STS' :: SP.S'P : ST.S'T :: CD^2 : ST^2$ .



58. If  $T$  be the centre, then, since the angles  $TSP$ ,  $TSA$  are equal,  $T$  lies on the tangent at  $A$ .

59. Let  $QL$  be the ordinate of  $Q$ .

Then  $QL : LS :: CN : NP$ ,

and  $QL : LS' :: CM : MD$ .

$QL^2 : SL \cdot SL' :: CM \cdot CN : PN \cdot DM :: AC^2 : BC^2$ ,  
or  $Q$  lies on an ellipse of which  $SS'$  is minor axis.

60. If  $P$  is the corresponding point on the ellipse, and  $SZ$  the perpendicular on the tangent to the circle,

$$\begin{aligned} SZ : AC &:: CT - CS : CT :: AC^2 - CS \cdot CN : AC^2 \\ &:: SP : AC; \\ \therefore SZ &= SP. \end{aligned}$$

61. The tangent at  $Q$  is parallel to the normal at  $P$ ;  
 $\therefore$  the tangent at  $P$  is parallel to the normal at  $Q$ .

62. If  $PQ P'Q'$  be the parallelogram, the angle  $DHE$  is the supplement of  $HPQ' + HQP'$ ;

that is, of  $SPQ + SQP$ ; that is, of  $DSE$ .

Hence  $S, H, D, E$  lie on a circle.

63. Let the line through  $C$  parallel to the tangent meet the directrices in  $Z, Z'$ .

Since the auxiliary circle is fixed,  $SY, S'Y'$  are fixed straight lines meeting  $ZZ'$  in fixed points  $y, y'$ .

And  $Cy \cdot CZ = CX \cdot CS = CA^2$ .

Therefore  $Z$  and  $Z'$  are fixed points.

64. The angle  $S'TZ = STY = SZY =$  complement of  $YZT$ .

Therefore  $YZ$  and  $S'T$  are at right angles.

65. If  $G$  be the centre of the circle,  $GL$  bisects  $SP$  at right angles.

Therefore  $SP$  is equal to the latus rectum.

66. If  $CZ$  be perpendicular to  $YY'$ , the perimeter of the quadrilateral is equal to  $SS'$  together with twice  $CZ + ZY$ , which is greatest when  $CZ = ZY$ , that is when  $SPS'$  is a right angle.

67. Draw  $SZ$  perpendicular to  $S'Z$  the straight line on which  $S'$  lies.

Let  $PS'P'$  be the chord parallel to  $SZ$ .

Produce  $PP'$  both ways to  $M$  and  $M'$ , so that

$$S'M = S'M' = AA'.$$

Then the lines drawn through  $M$  and  $M'$  perpendicular to  $SZ$  are fixed, and  $SP = PM$ ,  $SP' = P'M'$ .

Hence the ellipse will touch two parabolas having  $S$  for focus.

68. Let  $TQ$ ,  $TQ'$  be tangents,  $V$  the middle point of  $QQ'$ .

Then  $QV \cdot VQ' = CP^2 - CV^2 = CV \cdot VT$ .

Hence  $Q$ ,  $Q'$ ,  $C$ ,  $T$  lie on a circle.

69. Draw  $QM$  perpendicular to the minor axis.

Then  $QC^2 : AC^2 - CN^2 :: BC^2 : AC^2$ ,

or  $QC^2 : BC^2 :: AC^2 - CN^2 : AC^2$ .

Therefore  $BC^2 - CN^2 - QN^2 : CN^2 :: BC^2 : AC^2$ ,

or  $BC^2 - MC^2 : QM^2 :: AC^2 + BC^2 : AC^2$ .

Therefore  $Q$  lies on an ellipse of which  $BB'$  is minor axis.

70. If  $QM$  be the ordinate of  $Q$ ,

$$AM^2 : CN^2 :: QM^2 : PN^2 :: AM \cdot MA' : AC^2 - CN^2.$$

Therefore  $AM \cdot AA' : AM^2 :: AC^2 : CN^2$ ,

or  $2CN^2 = AC \cdot AM$ .

But  $AQ \cdot AO : CP^2 :: AM \cdot AC : CN^2$ ;

therefore  $AQ \cdot AO = 2CP^2$ .

$$71. \quad SP : SN :: SC : CQ :: SC : AC - QR.$$

Therefore  $SP \cdot AC = SP \cdot QR + SN \cdot SC.$

But  $SP : XS + SN :: SC : CA$

or  $SP \cdot AC = XS \cdot SC + SN \cdot SC.$

Therefore  $SP \cdot QR = XS \cdot SC = BC^2.$

72. If the tangent meet the tangent at  $A$  in  $T$ , and  $S'Y$ ,  $S'Z$  be perpendicular to  $TS$ , and the tangent,  $T, A, Y, S', Z$  lie on the circle of which  $S'T$  is diameter.

The angles  $Y TZ$ ,  $ATS'$  are equal since  $ATS$ ,  $S'TZ$  are equal. Art. 68.

Therefore the chords  $YZ$ ,  $AS'$  on which these angles stand are equal.

73. If  $P, Q, P', Q'$  be the parallelogram,  $p, q, p', q'$  the points of contact,  $pq, p'q'$  are parallel focal chords bisected by  $PCP'$ .

But  $QCQ'$  bisects  $pp'$ ,  $qq'$  and is therefore conjugate to  $PCP'$  and parallel to  $pq, p'q'$ .

Therefore  $CQ = CQ' = CA$

74. If  $T$  be the point from which the tangents are drawn,

$ST, S'T$  are perpendicular to  $TP', TP$  respectively.

Therefore  $SP, S'P'$  are both parallel to  $CT$ .

$$75. \quad CS^2 : CA^2 :: CG : CN :: CG \cdot CT : CN \cdot CT.$$

Therefore  $CG \cdot CT = CS^2.$

76. If  $PG$  be the normal at the point of contact,

$$CG \cdot CT = CS^2.$$

Therefore  $G$  is a fixed point and  $P$  lies on the circle of which  $GT$  is diameter.

77. Let the given straight line  $pq$  meet the axis  $t$ .

Let the tangents at  $p$  and  $q$  meet in  $Q$ .

Let  $CQ$  meet  $pq$  in  $V$  and the curve in  $P$

Through  $P, Q$  draw  $PG, QG'$  perpendicular to  $pq$ , meeting the transverse axis in  $G$  and  $G'$ .

Then  $CG' : CG :: CQ : CP :: CP : CV :: CT : Ct$ ;  
 $\therefore CG' . Ct = CG . CT = CS^2$ ,

or  $G'$  is a fixed point.

78. Draw  $SY$  perpendicular to the tangent; produce  $SY$  to  $L$  making  $LY = YS$ .

The point of intersection of the circles described with centres  $L$  and  $P'$ , and radii  $AA'$  and  $AA' - SP$  respectively will be the second focus.

79. Draw  $SY, SY'$  perpendicular to the given tangents.

The point of intersection of circles described with centres  $Y$  and  $Y'$ , and radius equal to  $CA$  will be the centre.

80. If  $OS$  be drawn perpendicular to  $PQ$ ,  $S$  will be one focus.

If  $SP, PS'$  be equally inclined to  $OP$  and  $SQ, QS'$  to  $OQ$ ,  $S'$  will be the other.

Bisect  $SS'$  in  $C$ , and take  $CA$  in  $SS'$  such that

$$2CA = SP + PS'.$$

If  $X$  be the foot of the directrix,

$$CX . CS = CA^2.$$

81.  $Qq : Aq :: PN : AN$ ,

and  $Rr : rA' :: PN : NA'$ .

Therefore  $Qq . Rr : Aq . A'r :: PN^2 : AN . NA'$   
 $:: BC^2 : AC^2 :: SL : AC$ .

Now  $Aq : qA' :: Aq^2 : Qq^2$ ,

and  $A'r : rA :: A'r^2 : Rr^2$ .

Therefore  $Aq . A'r : Ar . A'q :: AC^2 : SL^2$ .

82. By Ex. 75.  $CT : CS :: CS : CG$ ;

therefore  $TS : CS :: SG : CG$ .

But  $TY : PY :: TS : SG$ ;

$\therefore TY^2 : PY^2 :: CS^2 : CG^2 :: CT : CG :: TZ : PZ$ .

83. If  $O$  be the intersection of the lines

$$OC^2 = AC^2 + BC^2.$$

84.  $TP : TQ :: CP' : CQ'$ ,

and the angles  $PTQ, P'CQ'$  are equal.

Therefore  $PQ$  is parallel to  $P'Q'$ .

85.  $S, P, t, S'$  lie on a circle, and the triangles  $SCt, PYS$  are similar.

Therefore  $St : Ct :: SP : SY :: CD : BC$ ,

or  $St.PN : Ct.PN :: CD.BC : BC^2$ .

Therefore  $St.PN = CD.BC$ .

86. If the tangent at  $Q$  meet the minor axis in  $t'$ , the angle  $SQS' = St'S'$ , or  $t'$  is on the circle.

Now  $QM.Ct' = BC^2 = PN.Ct$ .

Therefore  $QM : PN :: Ct : Ct' :: Ct : Ct + St$   
 $:: BC : BC + CD$  by Ex. 85.

87. If  $S'Z$  be perpendicular to  $TY$ ,

the angle  $STY =$  complement of  $TZY'$ , (Ex. 64)

$=$  half supplement of  $YCY'$  the angle at the centre,

$= CYY'$ ; and  $STY = S'TY'$ .

88. The tangents at  $L$  and  $L'$  are perpendicular to the tangent at  $P$ , and therefore  $D$  and  $D'$  where they meet the tangent at  $P$  are on the director circle.

Now  $DL : DP :: D'L' : DP$ ;

therefore  $PQ$  bisects the angle  $LPL'$ .

Therefore  $LP + PL' =$  diameter of director circle.

89.  $AB, AE$  are equally inclined to  $BC$ ,

and  $AB^2 = AD.AE$ .

Therefore  $AB$  is a tangent.

90. If the tangent at  $P$  meet the tangent at  $A$  in  $T$ ,  $TS, TS'$  bisect the angles  $PSA, PS'A$ .

91. If the chords of intersection  $NO$ ,  $PQ$  meet in  $T$  and  $CD$ ,  $CE$ ,  $C'D'$ ,  $C'E'$  are parallel radii,

$$CD^2 : CE^2 :: TN.TO : TP.TQ :: C'D'^2 : C'E'^2.$$

92. If  $PN$  meet  $CD$  in  $K$ ,

$$PK : PQ :: SG : SP :: SA : AX,$$

and

$$PN.PK = PG.PF = BC^2.$$

Therefore  $PQ$  varies inversely as  $PN$ .

93. Draw perpendiculars  $SY$ ,  $CE$ ,  $S'Y'$ ,  $SZ$ ,  $CF$ ,  $S'Z'$  on tangents  $TP$ ,  $TQ$ ,

$$\text{then } CT^2 = CF^2 + TF^2 = CZ^2 + TZ.TZ' = CA'^2 + SY.S'Y' = CA'^2 + CB^2.$$

94. If the circle meet the minor axis in  $K$  and  $L$ , the tangents at  $P$  and  $Q$  meet either in  $K$  or  $L$ , see Ex. 15.

95. This problem is equivalent to Ex. 45.

96. Let  $CV$  bisecting the chord  $QSQ'$  meet the curve in  $P$  and directrix in  $T$ .

Let  $DCD'$  be the parallel diameter.

$$\text{Then } SR : SC :: CS - CR : CS :: CT - CV : CT$$

$$:: CS^2 - CG^2 : SC^2 :: SG.GS' : CS^2$$

$$:: SP.PS' : CA^2 :: CD^2 : CA^2 :: QQ'^2 : DD'^2$$

by Art. 76.

97. Let the tangent at  $Q$  meet  $PN$  in  $P'$  and the axis in  $U$ .

$$\text{Then } CT.CN = CA^2 = CM.CU;$$

$$\text{therefore } CT : CM :: CU : CN,$$

$$\text{or } TM : CM :: NU : CN.$$

$$\text{But } PN.QM : QM^2 :: TN : TM,$$

$$\text{or } PN.QM : CM.MU :: CN.NT : CM.NU.$$

$$\text{Hence } PN.QM : CN.NT :: CM.QM : CM.P'N$$

$$:: QM^2 : P'N.QM.$$

$$\text{Now } PN^2 : CN.NT :: BC^2 : AC^2 :: QM^2 : Q'M^2.$$

$$\text{Hence } PN.Q'M : PN^2 :: Q'M^2 : P'N.QM.$$



Therefore  $P'N : PN :: AC : BC$ ,  
or  $P'$  is on the auxiliary circle.

98. The diameter bisecting  $PQ$  is fixed, hence  $V$  the centre of the circle, is a fixed point.

$VM$  bisecting  $RS$  at right angles, is a fixed straight line ;

$PQ$  and  $RS$  are equally inclined to the axis ;

$\therefore CM$  and  $CV$  are equally inclined to the axis.

Therefore  $M$  is a fixed point, and  $RS$  a fixed straight line.

99. The angle  $BSC$

$$\begin{aligned} &= BAC + SBA + SCA = BAC + HBC + HCB \\ &= BAC + \text{supplement of } BHC. \end{aligned}$$

Hence if  $BHC$  is constant,  $BSC$  will be constant.

100. The angles  $SPT$ ,  $HPt$  are each equal to  $SQH$  ;

$$\begin{aligned} \text{also} \quad STP &= tQS - PtH \\ &= HPt - PtH = PHt. \end{aligned}$$

Therefore  $TP : SP :: HP : Pt$ ,

or  $TP \cdot Pt = SP \cdot HP = CD^2$ .

Therefore  $CT$ ,  $Ct$  are conjugate.

101.  $CT$  bisects  $PQ$  and is parallel to  $SP$  ;

$\therefore T$  is the foot of the perpendicular from  $S'$  on  $PT$ .

See Cor. (3), Art. (66).

102.  $SC : CY$  is a given ratio and  $SY$  is fixed.

103. Take  $p$  a point near and let the focal chord  $p'Sq'$  meet  $pq$  in  $O$  ;

$$\begin{aligned} PQ : p'q' &:: pO \cdot Oq : p'O \cdot Oq' :: ST \cdot Oq : Sp' \cdot Oq' \\ &:: ST \cdot pq : Sp \cdot p'q' \text{ ultimately.} \end{aligned}$$



104. Dropping perpendiculars from the focus on the sides, their feet are the middle points, and, as they lie on a circle, form a rectangle; the diagonals, intersecting in  $H$ , are therefore at right angles, and  $SAD$  can be proved equal to  $HAB$ .

105. If  $CD$ ,  $CP$  meet the directrix in  $E$  and  $G$ ,  $ES$  is perpendicular to the chord of contact of tangents from  $E$ , which is parallel to  $CP$ .

106. If  $CP$ ,  $DC$  meet the tangent at  $A$  in  $Q$  and  $R$ , prove that  $AQ \cdot AR = BC^2 = AS \cdot AS'$ .

Then  $QSA = ARS'$ , and  $QRA = AQS'$ .

## CHAPTER IV.

### THE HYPERBOLA.

1. If the circle whose centre is  $P$  touch the circles whose centres are  $S$  and  $H$  in  $Q$  and  $R$ ,

$$SP \sim HP = SQ \sim HR.$$

Therefore  $P$  lies on an hyperbola of which  $S$  and  $H$  are foci.

$$2. \quad SD^2 = BC^2 = CS^2 - CA^2 = CS^2 - CX \cdot CS = CS \cdot SX.$$

Therefore the triangles  $SCD$ ,  $SDX$  are similar.

3. If the straight line meet the curve in  $P$  and the directrix in  $F$ ,

$$SF : SX :: CS : CA :: SA : AX :: SR : SX.$$

Therefore  $SF = SR.$

Draw  $PK$  perpendicular to the directrix.

Then  $PF : PK :: SC : CA :: SP : PK.$

Therefore  $SP = PF.$

4. Draw  $SD$ ,  $SD'$  perpendicular to the asymptotes.

Then  $DD'$  is the directrix.

5. If the asymptote meet the directrix in  $D$ , then  $DS$  drawn at right angles to  $CD$  meets the axis in the focus.

6. If  $PK, QL$  be the perpendiculars from the given points on the directrix  $PS-SQ=PK-QL$  which is constant.

Therefore  $S$  lies on an hyperbola of which  $P$  and  $Q$  are foci.

7. If the circle inscribed in the triangle  $ABC$  touch the sides in  $D, E, F$ ;  $B, C, D$  being given,

$$BA-CA=BF-EC=BD-DC.$$

Hence  $A$  lies on an hyperbola of which  $B$  and  $C$  are foci and  $D$  a vertex.

$$8. \quad PN : Pg :: SY : SP :: BC : CD$$

$$:: PF : AC :: AC : Pg.$$

Therefore

$$PN=AC.$$

9. Draw  $CD$  parallel to the given line and  $CP$  parallel to the tangent at  $D$ .

Then the tangent at  $P$  is parallel to  $CD$  and the given line.

10. Let  $A'P$  and  $P'A$  meet in  $Q$ , and draw the ordinate  $QM$ .

$$\text{Then} \quad QM : A'M :: PN : NA',$$

$$\text{and} \quad QM : AM :: P'N : NA.$$

$$\text{Therefore} \quad QM^2 : AM.MA' :: PN^2 : AN.NA'$$

$$:: BC^2 : AC^2,$$

or  $Q$  lies on an hyperbola having the same axes.

11. Let the tangent at  $P$ ,  $AP$  and  $A'P$  meet the minor axis in  $t, E$  and  $E'$ .

$$\text{Then} \quad CE : PN :: CA : AN :: CA.A'N : AN.NA',$$

$$\text{and} \quad CE' : PN :: CA.AN :: AN.NA'.$$

$$\text{Hence} \quad CE-CE' : PN :: 2CA^2 :: AN.NA'.$$

$$\text{Now} \quad PN^2 : AN.NA' :: PN.Ct : AC^2;$$

therefore

$$CE-CE'=2Ct.$$

Therefore  $Pt$  bisects every line perpendicular to  $AA'$  terminated by  $A'P, AP$ .

12.  $SPT$  is an isosceles triangle since  $pST$  is parallel to  $S'P$ .

Therefore

$$SP = ST.$$

13. Draw  $SD$  perpendicular to the asymptote and  $SK$  parallel to it.

If  $TS$  bisect the angle  $PSK$ ,  $T$  being on the asymptote,  $TP$  is the tangent at  $P$ . Draw  $SY$  perpendicular to it.

Then  $CM$  which bisects  $DY$  at right angles will meet the asymptote in the centre  $C$ .  $DX$  drawn perpendicular to  $CS$  will be directrix.

14. If the tangent at  $P$  meet the tangents at  $A$  and  $A'$  in  $Z$  and  $Z'$ ,  $ZS$  and  $Z'S$  are the internal and external bisectors of the angle  $ASP$ .

Hence the foci lie on a circle of which  $ZZ'$  is diameter.

15. Draw  $PK$  perpendicular to the directrix and  $DS$  at right angles to the asymptote.

Draw  $cs$  at right angles to the directrix meeting it in  $x$ .

With centre  $P$  and radius  $PS$  such that  $SP : PK :: cs : cD$ , describe a circle meeting  $Ds$  in  $S$ .

Then  $S$  is the focus.

16. Draw  $Qq'$ ,  $Pp$  and  $Rr$ ,  $Qq$  parallel to the asymptotes.

Then  $CP : CQ :: Cp : Cq' :: Cq : Cr :: CQ : CR$ .

$$17. \quad QC^2 - CB^2 : CN^2 - CA^2 :: BC^2 : AC^2 \\ :: PN^2 : CN^2 - CA^2.$$

Therefore

$$PN^2 = QB \cdot QB'.$$

$$18. \quad DN : NA :: QM : MA \text{ and } EN : NA' \\ :: QM : MA'.$$

$$\text{Therefore } ND \cdot NE : AN \cdot NA' :: QM^2 : AM \cdot MA' \\ :: PN^2 : AN \cdot NA',$$

or

$$PN^2 = ND \cdot NE.$$

19.  $A, A', Y, Z$  lie on the auxiliary circle.

Therefore

$$AT \cdot TA' = YT \cdot TZ.$$

20. If the tangent at  $P$  meet the asymptotes in  $L$  and  $L'$ ,  $CD=PL=PL'$ ; therefore  $Q$  divides  $LL'$  in a constant ratio.

Draw  $QH$ .  $QK$  parallel to the asymptotes.

Then  $QH \cdot QK$  varies as  $CL \cdot CL'$  and is therefore constant.

Therefore  $Q$  lies on an hyperbola having the same asymptotes.

21. This is equivalent to the preceding.

22. Since  $SK=S'K$ ,  $K$  lies on the circle passing through  $S$ ,  $S'$  and  $P$ , and since  $KPt$  is a right angle,  $t$  lies on the same circle.

Therefore  $GK : S'K :: SG : SP :: SA : AX$ ,

and  $St : tK :: SY : SP :: BC : CD$ .

23. Let  $P, P'$  be the points of trisection of the arc  $SS'$  and let  $XM$  bisect  $SS'$  at right angles, then  $SP=2PM$  and  $S'P'=2P'M$ .

Hence  $P$  and  $P'$  lie on hyperbolas of which  $S$  and  $S'$  are foci and  $XM$  directrix.

If  $C, C'$  be the centres  $CS=4CX$  and  $C'S'=4C'X$ .

Therefore  $C$  and  $C'$  are the points of trisection of the chord  $SS'$ .

24. Draw  $SZ$  parallel to the asymptote: the angle  $STQ=TSZ=TSP$ .

Therefore  $SQ=QT$ .

25. Since the hyperbolas have the same asymptotes the ratios  $CS : BC : CA$  are constant.

Let  $NP$  be the fixed line parallel to an asymptote, and  $PQ$  proportional to an axis.

Then  $PQ^2$  varies as  $CS^2$ , that is, as  $CN \cdot NP$ , that is, as  $NP$ .

Hence  $Q$  lies on a parabola having  $NP$  for a diameter.

$$26. \quad PY.PY' = AC^2 - CP^2 = CD^2 - BC^2 = CS^2 \\ - \left( \frac{SP - S'P}{2} \right)^2 = CS^2 - CA'^2 = CB'^2.$$

27. Let  $TP$  meet the other asymptote in  $T'$ , then  $PT = PT'$ .

Therefore  $PQ = R'P = QR$ .

28 Draw  $OrR$  parallel to  $PQ$ , meeting the ellipse and hyperbola in  $r$  and  $R$ .

Let  $Oa, Ob$  be the axes, then since  $OP, Or$  are conjugate in the ellipse, and  $OQ, OR$  in the hyperbola, if  $PN, QM, rl, RL$  be the ordinates,

$$ON.Ob = rl.Oa; PN.Oa = Ol.Ob; QM.Oa \\ = OL.Ob; OM.Ob = RL.Oa.$$

Therefore  $PN : ON :: QM : OM$

since  $rl : Ol :: RL : OL,$

or  $OP$  and  $OQ$  are equally inclined to the axes.

29. Through  $S$  draw  $SC$  parallel to the bisector of the angle between the asymptotes meeting the asymptote which is given in position in  $C$ .

Draw  $SD$  perpendicular to that asymptote, and  $DX$  to  $CS$ . Then if  $A$  be taken in  $CS$  such that  $CA^2 = CX.CS$ ,  $A$  is vertex.

30. If the tangents at  $P$  and  $Q$  meet in  $T$ , then since the perpendiculars from  $T$  on  $SP, SQ, HP, HQ$  are all equal, a circle can be described with centre  $T$  to touch  $SP, SQ, HP$  and  $HQ$ .

31. If  $CL, CL'$  be the asymptotes,  $S$  will lie in the bisector of the angle  $LCL'$ .

Draw  $PL, PL'$  parallel to the asymptotes to meet them in  $L$  and  $L'$ ;

and take  $S$  in  $CS$  such that  $CS^2 = 4CL.CL'$ ,

then  $S$  is a focus.



32. If the conjugate diameters  $PCP'$ ,  $DCD'$  be given, complete the parallelogram  $LML'M'$  formed by the tangents at  $D$ ,  $P$ ,  $D'$  and  $P'$ .

The diagonals  $LL'$ ,  $MM'$  are the asymptotes and the axes bisect the angles  $LCM$ ,  $LCM'$ .

33. Let  $QT$  and  $RQ$  meet the asymptotes in  $L$  and  $M$ .

$$\begin{aligned} \text{Then } QL : PH :: RH : TL :: CR : CT \\ :: RM : TK :: PK : QM; \end{aligned}$$

$$\text{therefore } QL \cdot QM = PH \cdot PK,$$

or  $Q$  is on the curve.

34. Let  $CL$  bisecting the angle  $ACB'$  meet  $PN$  in  $L$ , draw  $QM$  parallel to  $LC$ .

Then  $CL$  is proportional to  $CN$ ;

therefore  $CL \cdot CM$  is proportional to  $CN \cdot NT + CN^2$ , that is to  $CA^2$ .

Hence  $Q$  lies on an hyperbola of which  $CL$  and  $CB$  are asymptotes.

35. Draw  $SY$  perpendicular to the tangent and produce it to  $Z$  making  $YZ = SY$ .

Then if  $Q$  be the point of contact, and  $P$  the fixed point,

$$HP - PS = HQ - QS = HZ;$$

$$\text{therefore } HP - HZ = PS,$$

or the locus of  $H$  is an hyperbola of which  $P$  and  $Z$  are foci.

36. If  $PT$ ,  $Pt$  the tangents to the ellipse and hyperbola meet  $BC$  in  $T$  and  $t$ , then since the curves have the same conjugate axis, for  $CA^2 = CS^2 + CB^2$

$$Ct \cdot PN = BC^2 = CT \cdot PN,$$

$$\text{or } CT = Ct.$$

37. This problem is the converse of Ex. 3.



38. If  $G$  be the point of intersection

$$CG = \frac{2}{3} CP,$$

or  $G$  lies on an hyperbola having the same asymptotes.

39. The angle  $CYY' = S'PY' = S'NY'$  since  $S', P, N$  and  $Y'$  lie on a circle.

Therefore  $Y, Y', C$  and  $N$  lie on a circle.

40. If  $SY$  meet  $S'P$  in  $Z$ ,  $SY = YZ$ ; therefore  $S'Y$  bisects  $PG$ .

Similarly  $SY'$  bisects  $PG$ .

41.  $BC^2 : AC^2 :: NG : CN :: CT.NG : CN.CT$ ;  
therefore  $CT.NG = BC^2$ .

42. The angle  $STP = TS'P + S'PT = TS'P + SPT$   
 $= PS'S + SS't = \text{supplement of } PSt$ .

43.  $Pt.PT = CD^2 = SP.S'P$ ,  
or  $SP : PT :: Pt : S'P$ ;

and the angles  $SPT$  and  $tPS'$  being equal, the triangles  $SPT, tPS'$  are similar.

44. The circles  $SCE, S'CE'$  stand upon equal chords  $SC, S'C$  and contain equal angles  $SEC, S'E'C$ , since  $CE$  is parallel to the bisector of  $SPS'$ .

45. If the tangent at  $P$  meet the tangent at  $A$ , the vertex of the branch on which  $P$  lies, in  $T$ ,  $T$  is the centre of the circle inscribed in the triangle  $SPS'$ , since  $TS, TS'$  bisect the angles  $ASP, AS'P$ .

46.  $CT^2 : CA :: CA : CN :: CP : CQ$ , or  $AQ$  is parallel to  $PT$ .

47.  $CE : CA :: CS : CA$ ;  
therefore  $CE = CS$ ; but  $CD = CA$ .  
Therefore  $AD$  and  $SE$  are parallel.

48. If  $E$  be the centre of the circle and  $EK$  its radius,  
 $EK : CE :: BC : SC$ ,

$$\begin{aligned} \text{or} \quad EK : CA &:: BC : SC + BC \\ &:: (SC - BC) BC : CA^2. \end{aligned}$$

$$\text{And} \quad SR' : CS :: BC : AC :: SR : BC;$$

$$\text{therefore} \quad RR' : SR :: CS - BC : BC,$$

$$\text{or} \quad RR' : CS - BC :: SR : BC :: BC : CA.$$

$$\text{Therefore} \quad EK = RR'.$$

$$49. \quad PM = PL;$$

$$\text{therefore} \quad GL = GM.$$

50. If  $Ca$ ,  $Cb$  be the conjugate diameters and one hyperbola touch the ellipse in  $P$ , the tangent at  $P$  will meet  $Ca$ ,  $Cb$  in  $T$ ,  $t$ , such that  $TP = Pt = CD$ .

Hence  $PD$  is bisected by  $Ct$ ,

and  $tD$  touches the other hyperbola and is parallel to  $CP$ .

51. If  $LL'$  and  $MM'$  be the tangents,

$$CL : CM :: CM' : CL',$$

or  $LM'$  and  $L'M$  are parallel.

52. If the tangent at  $P$  meet the tangents at  $A$  and  $A'$  in  $F$  and  $F'$  and  $QM$  be the ordinate of the centre of the circle,

$$QM : MS :: SA : AF,$$

$$\text{and} \quad QM : MS' :: S'A' : A'F'.$$

$$\text{Hence } QM^2 : SM \cdot MS' :: SA^2 : AF \cdot A'F' :: SA^2 : BC^2$$

(Art. 126).

Hence the locus of  $Q$  is an hyperbola of which  $S$  and  $S'$  are vertices.

53. If  $PM$  be perpendicular to the directrix,

$$PK : PM :: CS : CA :: SA : AX :: SP : PM,$$

$$\text{or} \quad PK = SP.$$

54. Let  $PD$  meet an asymptote in  $n$ , draw  $Pl$ ,  $Dm$  parallel to  $Cn$ .

Then

$$Dm \cdot Dn = Pn \cdot Pl;$$

therefore

$$Dn = Pn.$$

Therefore if  $LPL'$  is tangent at  $P$ ,  $LD$  is tangent at  $D$ , and  $CP$ ,  $CD$  are conjugate.

55. If  $QR$  meet the asymptotes in  $q$  and  $r$ ,  $qQ = rR$ ; therefore if  $EPe$  be the tangent at  $P$ ,

$$CL : CN :: qQ : Pe :: PE : qR :: CN : CM.$$

56. If the circle intersect the axis in  $b$ ,  $B$ ,

$$CB \cdot Cb = CS^2$$

or

$$CB^2 + CB \cdot Bb = CA^2 + CB^2;$$

therefore

$$CB \cdot Bb = CA^2.$$

57. Let the straight line  $q'Q'APQq$  meet the asymptotes in  $Q'$ ,  $Q$ .

Draw  $RCR'$  parallel to  $AP$  terminated by  $A'q$ ,  $A'q'$ .

Then

$$PQ' = AQ = CR = Qq,$$

and

$$Q'q' = CR' = AQ' = PQ;$$

therefore

$$Pq' = Pq.$$

58.  $T$  is the centre of the circle inscribed in the triangle  $PS'Q$ , therefore the difference between  $PTQ$  and half  $PS'Q$  is a right angle.

59. Draw  $CD$ ,  $CE$  parallel to  $OA$ ,  $OB$  and  $PH$ ,  $PK$  parallel to and terminated by  $CE$ ,  $CD$ .

Then  $PH : OD :: CK : CE :: CP : CA :: CB : CP$   
 $:: CD : CH :: OE : PK;$

therefore

$$PH \cdot PK = OD \cdot OE,$$

or  $P$  lies on an hyperbola having  $CD$ ,  $CE$  for asymptotes.

60. Draw  $PH$ ,  $PK$ ,  $QH'$ ,  $QK'$  parallel to the asymptotes.

Then  $PL : QM :: PH : QH' :: QK' : PK :: QN : PR$ ,

or

$$PL \cdot PR = QM \cdot QN.$$

61. If  $TK$ ,  $TN$  be perpendicular to the directrix and  $SP$ ,  $TK=SN$ .

Therefore  $ST : TK :: ST : SN$  a constant ratio,  
and the angle between the asymptotes is double  $PST$ , that is, double the external angle between the tangents.

$$62. \quad Q'V^2 - RV^2 = CD^2 = RV^2 - QV^2,$$

$$\text{or} \quad QV^2 + Q'V^2 = 2RV^2.$$

$$\text{Again} \quad CT \cdot CV = CP^2 = CV \cdot CT';$$

$$\text{hence} \quad CT = CT'$$

63. If  $V$  be the middle point of  $PQ$ , then since  $R$ ,  $V$  are the middle points of  $LRL'$  and  $LPQL$ ,  $RV$  is parallel to the asymptote  $CL'$ .

$$\text{Hence} \quad PM + QN = 2RE.$$

64. If  $TP$ ,  $TQ$  be the tangents,  $PTQ$ ,  $STS'$  have the same bisector which passes through the point where the circle meets  $BCB'$ .

65. The tangents at  $P$  and  $Q$  intersect in  $t$  on the circle and  $BCB'$ .

Hence the angle  $PtQ = PSQ$ .

66. The angle  $PNY = PSY = PS'Y' =$  supplement of  $PNY'$ .

67. The triangle  $YCY'$  is greatest when  $YCY'$  or  $SPS'$  is a right angle.

In that case  $PT$  meets  $BC$  in  $t$  such that  $Ct = CS$ ;

$$\text{therefore} \quad CS \cdot PN = Ct \cdot PN = BC^2.$$

$$68. \quad \text{The triangle } SPS' : \text{triangle } StS' :: SP \cdot PS' : St^2 \\ :: CD^2 : St^2.$$

69.  $S'P$  is parallel to  $CY$  and  $S'Q$  to  $CZ$ .

Therefore  $S'T$  is parallel to bisector of  $YCZ$  and is perpendicular to  $YZ$ .

70. If  $G$  be the centre of the circle,  $GL$  bisects  $SP$ ;  
therefore  $SP = 2PL = 2SR$ .

71. The tangent at  $Q$  is parallel to the normal at  $P$ , therefore the tangent at  $P$  is parallel to the normal at  $Q$ , or  $CP$  is conjugate to normal at  $Q$ .

72. If  $Y$  be the point from which the tangents are drawn,  $SP$  and  $S'P'$  are both parallel to  $CY$ .

73.  $SC^2 : AC^2 :: CG : CN :: CG \cdot CT : CN \cdot CT$ ,  
or  $CG \cdot CT = SC^2$ .

74. By Ex. 73,  $G$  the foot of the normal is a fixed point;

therefore  $P$  lies on the circle of which  $TG$  is diameter.

75. If  $TP$ ,  $TQ$  be the tangents,  $CT$  will bisect  $PQ$  in  $V$ ,  
and  $CT \cdot CV = CT^2$ ,

or  $PQ$  is a tangent at  $V$ .

76. Let  $GQ$  meet the conjugate in  $G'$ .  
Then  $QG' : QG :: CN : NG :: AC^2 : BC^2$ .  
Therefore, by Art. 111,  $QG'$  is normal at  $Q$ .

77. If  $PM$  be drawn perpendicular to the directrix of the parabola the angle  $PTQ = SPT - SQT = \text{half } SPM - \text{half } SQS' = \text{half } SS'Q$ .

78. If  $abcd$  be the quadrilateral and  $S$  lie on the circle the angle  $Hcd = Scb = Sab = Had$ ,  
or  $H$  is on the circle.

79. If  $PP'$  be the chord of contact and  $CV$  bisect  $PP'$  then  $CV$ ,  $PP'$  are parallel to a pair of conjugate diameters in both conics.

Hence if from a common point  $Q$ , a double ordinate  $QVQ'$  be drawn parallel to  $PP'$ ,  $Q'$  must lie on both curves.

Similarly  $RR'$  the line joining the other two common points is parallel to  $PP'$ .

80. If  $SD, SD'$  are perpendiculars from the common focus on the asymptotes,  $DD'$  is the tangent at the vertex of  $P$  and a directrix of  $H$ .

If  $P$  be a common point, and  $PM$  perpendicular to  $DD'$ ,

$$SP : PM :: SC : CA,$$

but  $SP = PM + SX$ .

Therefore  $SP : SX :: CS : CS - CA :: CS : AS$ .

Hence  $AS \cdot SP = SX \cdot CS = BC^2 = AS \cdot SA'$ ,

or  $SP = A'S$ .

Therefore  $A'P$  touches the parabola at  $P$ .

81. With centre  $P$ , the given point and radius of the given length describe a circle meeting the other asymptote in  $p$ .

Then  $pPQq$  is the line required.

82. Let  $CB, CA$  be semiaxes of the ellipse,  $Ca, Cb$  of the hyperbola.

Let  $PN$  meet the asymptote in  $Q$ ,

then  $QN^2 : CN^2 :: Cb^2 : Ca^2$ ,

or  $QN^2 + CN^2 : Ca^2 + Cb^2 :: CN^2 : Ca^2$ ;

but  $SP + S'P = 2CA$ ,

and  $SP - S'P = 2Ca$ .

Hence  $4CA \cdot Ca = SP^2 - S'P^2 = SN^2 - S'N^2 = 4CN \cdot CS$ ;

therefore  $CA^2 : CS^2 :: CN^2 : Ca^2 :: QN^2 + CN^2 : Ca^2 + Cb^2 :: CQ^2 : CS^2$ .

Therefore  $Q$  lies on the auxiliary circle of the ellipse.

83. Let  $Q$  be a common point.

Then  $SQ - QH = AA'$  and  $SQ - QP = SP - 2PH = AA' - PH$ .

Therefore  $QP = QH + PH$ ,

or  $Q$  must be the other extremity of the focal chord  $PH$ .



84. If  $A'K$  meet the directrix in  $F$ , then,  
 since  $SA' = 2A'X$ ,  
 $FA'S$  is an isosceles triangle and  $FS$  is parallel to  $KD$ .

Also  $A'F : FP :: A'X : XN :: A'S : SP$   
 or  $FS$  bisects the angle  $A'SP$ ;

therefore if  $SP$  and  $DK$  meet in  $Q$ ,  $QSD$  is an isosceles triangle.

Therefore  $Q$  lies on the circle of which  $A'D$  is diameter.

85. This problem is a particular case of Ex. 61.

86.  $PL \cdot PL' = PL^2 = CD^2 = PG \cdot Pg$ ;  
 therefore  $G, g, L, L'$  lie on a circle of which  $Gg$  is diameter.

$C$  is on this circle since  $GCg$  is a right angle.

The radius of this circle varies as  $Gg$  and therefore as  $CD$  and therefore inversely as the perpendicular from  $C$  on  $LL'$ .

87. If  $PCP', DCD'$  be conjugate diameters and  $Q$  any point on the curve,

$$QP^2 + QP'^2 = 2CP^2 + 2CQ^2; \quad QD^2 + QD'^2 = 2CD^2 + 2CQ^2.$$

Therefore

$$QP^2 + QP'^2 - QD^2 - QD'^2 = 2CP^2 - 2CD^2 = 2AC^2 - 2BC^2.$$

88. If  $S'L', S'M'$  be drawn parallel to the asymptotes  $LS', MS'$  bisect the angles  $PS'L', PS'M'$ .

Hence  $LS'M$  = half the angle between the asymptotes.

89. If  $PT$  meets the tangent at  $A$  in  $V$ ,  $VS$  bisects the angle  $ASP$ ;

therefore  $SP : ST :: PV : VT :: AN : AT$ .

90. If  $P$  is a point of intersection, let the tangent and normal of the ellipse at  $P$  meet the transverse axis in  $T$  and  $G$ , and the conjugate axis in  $t$  and  $g$ .



Then,  $PT$  being the normal of the hyperbola, the semi-axes of which are  $A'C$  and  $B'C$ ,

$$\begin{aligned} CT : CN &:: SC^2 : A'C^2, & \text{Art. 111,} \\ \therefore AC^2 : CN^2 &:: SC : A'C; \\ \therefore CN.SC &= AC.A'C. \end{aligned}$$

Again,  $Pg$  being the normal of the ellipse,

$$Cg : PN :: SC^2 : BC^2, \quad \text{Art. 72,}$$

and

$$Cg.PN = B'C^2,$$

$$\therefore B'C^2 : PN^2 :: SC^2 : BC^2$$

and

$$PN.SC = BC.B'C.$$

Hence, if  $PN$  meet the asymptote in  $Q$ ,

$$QN : CN :: B'C : A'C,$$

and it is easily deduced that

$$QN : PN :: AC : BC.$$

91. Let  $ABCD$  be the quadrilateral,  $A$ ,  $B$ , and  $C$  being fixed points.

Then

$$AB + CD = BC + AD,$$

or

$$CD - DA = CB - AB.$$

Hence  $D$  lies on an hyperbola of which  $A$  and  $C$  are foci.

92. Since  $Q$ ,  $S$ ,  $C$ ,  $t$  lie on a circle, the angle

$$tQC = tSS' = SPt,$$

hence  $CQ$  is parallel to  $SY$  and  $CY$ ,  $SQ$  are equally inclined to  $SY$ ;

therefore

$$SQ = CY = CA.$$

93. Draw  $SY$  perpendicular to the tangent and produce to  $Z$  making  $SY = YZ$ .

Then if  $P$  be the point of contact  $HZ = HP - SP = AA'$ .

Hence the locus of  $H$  is a circle of which  $Z$  is centre.

94.  $RS$  and  $VS'$  bisect the angles  $PSQ$  and  $PS'Q$ ; let  $QS$ ,  $S'P$  meet in  $Z$ .

Then  $RSP + VS'Q = \text{half } PSQ + \text{half } PS'Q = \text{half } QSP$   
 $+ \text{half } SZS' - \text{half } SQS' = QSP + \text{half } SPS' - \text{half } SQS'$   
 $= QTP + TQS - SQT = PTQ.$

95.  $CgP$  is an isosceles triangle, and the angle

$$CGt = CPT = TCP;$$

therefore  $PG = Ct$  and  $CD^2 = PG \cdot Pg = Cg \cdot Ct = CS^2.$

96. Since the asymptote  $CD$  bisects  $BA$ ,  $CD$  is parallel to the axis of the parabola and  $BA$  is parallel to the other asymptote.

If  $QPVP'Q'$  parallel to  $BA$  meet  $CD$  in  $V$ ,

$$QV = VQ' \text{ and } PV = VP';$$

therefore

$$QP = Q'P'.$$

97. Let  $EL$  be the ordinate of  $E$ , and draw  $EF$  perpendicular to  $PN$ .

Then,  $CD$  being conjugate to  $CP$ , the triangles  $CDM$  and  $PFE$  are similar and equal.

$$\therefore CL = CN + EF = CN + DM,$$

$$\therefore CN : CL :: AC : AC + BC,$$

and similarly  $EL : PN :: BC : BC - AC;$

$$\therefore EL^2 : (BC - AC)^2 :: PN^2 : BC^2$$

$$:: CN^2 - AC^2 : AC^2$$

$$:: CL^2 - (AC + BC)^2 : (AC + BC)^2.$$

98. If  $PM$  be drawn from the centre perpendicular to  $BC$

$AP : PM :: PC : PM$ , a constant ratio;

therefore  $P$  lies on an hyperbola of which  $A$  is focus and  $BC$  directrix.

If  $S$  be the other focus and  $SP$  meet the circle in  $Q$

$$SQ = SP - PA = \text{constant},$$

or, the envelope is a circle of which  $S$  is centre.

99. The conics will be confocal having their foci  $H$  and  $H'$  on  $PG$ ,

such that  $PH^2 = PT \cdot Pt = CD^2.$

For their locus see Ex. 9 on the ellipse.

100. If  $SY, SZ, S'Y', S'Z'$  be perpendiculars on tangents at right angles

$$CT^2 - CA^2 = TY \cdot TY' = SZ \cdot SZ' = CB'^2.$$

If  $SYZ, S'Y'Z'$  are perpendicular to parallel tangents and  $CWW'$  be the perpendicular through the centre

$$2CW = SY + S'Y'; \quad 2CW' = SZ - S'Z',$$

and

$$SY - SZ = S'Y' + S'Z';$$

$$\therefore 4CW'^2 + 4CB'^2 = 4CW^2 - 4CB^2.$$

Hence

$$CW^2 - CW'^2 = CB^2 + CB'^2.$$

101. The chord  $QR$  is inclined to the axis at the same angle as the tangent at  $P$  and is therefore always parallel to a fixed line.

102.  $TP$  and the asymptote subtend equal angles at  $S'$ ;

$$\therefore PS'T = S'TC = STP.$$

$$103. \quad SF^2 = FX^2 + SX^2 = CF^2 - CX^2 + SX^2$$

$$= CF^2 + CS^2 - 2CS \cdot CX$$

$$= CF^2 + CS^2 - 2CA^2 = CF^2 - CA^2 + CB^2$$

$$= \text{square of tangent from } F$$

$$= FA \cdot FB.$$

104. If  $S$  be the focus of the ellipse and  $S'$  of the hyperbola,

$$CS : CA :: CA : CS';$$

$\therefore S$  and  $S'$  coincide with the feet of the directrices.

The relation,  $CN \cdot CT = CA^2$ , proves that  $S$  and  $S'$  are the feet of the ordinates, and the relation,  $Ct \cdot PN = BC^2$ , proves that  $t$  and  $t'$  are on the auxiliary circle.

Also  $Ct' : CS = AC : CS = CS' : Ct$ ;

$\therefore$  the tangent intersects at right angles.

105. For  $PG \cdot Pg = CD^2$ , Art. 123,  $= PL^2 - PL \cdot PL'$ , and the diameter of the circle is  $Gg$ , which varies as  $CD$ , Art. 123, and therefore inversely as  $CY$ .

## CHAPTER V.

### THE RECTANGULAR HYPERBOLA.

1. THE angle  $PCL = CLP =$  complement of  $LCY$ .

2.  $QV^2 = VP \cdot Vp$ , hence  $VQ$  touches at  $Q$  the circle  $QPp$ .

3.  $LP = PM = CD = PG = Pg$ .

Hence  $LGM$  is a right angle.

4. If  $LM$  be the straight line, and  $C$  be the corner of the square,  $CL \cdot CM$  is constant, hence  $LM$  touches an hyperbola of which  $CL, CM$  are asymptotes.

5. Let  $AP, A'P'$  meet in  $Q$ , and draw the ordinate  $QM$ .

Then  $QM : MA :: PN : NA$ ,

and  $QM : MA' :: P'N : NA'$ .

Hence  $QM^2 : AM \cdot MA' :: PN^2 : AN \cdot NA'$ ;

therefore  $QM^2 = AM \cdot MA'$ .

Hence  $Q$  lies on a rectangular hyperbola having  $AA'$  for transverse axis.

6. If  $P, P'$  be joined to  $Q$  meeting an asymptote in  $R$  and  $R'$ ,  
the angle

$$QRL = CLP - QPL = LCP - PP'Q = CR'P' = QR'L.$$

7. Produce  $LP$  to  $M$  making  $PM=PL$ , then if  $MC$  be drawn perpendicular to the given asymptote  $CL$ ,  $C$  is the centre. In  $CS$  the bisector of the angle  $LCM$  take  $S$  such that  $CS$  is a mean proportional between  $CL$  and  $CM$ .

Then  $S$  is focus, and  $X$  the middle point of  $CS$  is the foot of the directrix.

8. The angle  $DCL=PCL$ ,  
and  $D'CL=P'CL$ ;  
hence  $DCD'=PCP'$ .

9. A diameter is a mean proportional between the parallel focal chord and  $AA'$ , therefore focal chords parallel to conjugate diameters are equal.

10. As in the preceding, focal chords at right angles are equal, since diameters at right angles are equal.

11. If  $CD, Cd$  be conjugate to  $CP$  in the ellipse and hyperbola,

$$CD^2=SP \cdot PS'=Cd^2=CP^2.$$

12.  $PN=CM$ ;  $DM=CN$ ;  
and  $CD=CP$ .

13.  $SP \cdot PS'=CD^2=CP^2.$

14.  $QV^2=CV^2-CP^2=CV^2-CT \cdot CV=CV \cdot VT.$

Hence  $QV$  touches the circle  $CTQ$ .

15. If  $D$  be the intersection of tangents at  $A$  and  $B$ ,

$$CD^2=AC^2+BC^2=SC^2.$$

Hence  $D$  lies on the circle of which  $SS'$  is diameter.

16. If  $LPM, G'PG$  be the tangent and normal to one,

$$LP=PM=CD=CP=PG=PG';$$

therefore  $GG'$  is the tangent to the second hyperbola.

17. The angle  $CRT=CQT+RTQ=2CLQ+LTL'$   
 $=CLM+CL'T=CLR'+L'CQ'=TR'Q'.$

Hence  $C, T, R$  and  $R'$  lie on a circle.

$$18. \quad QR^2 = CN^2 = CA^2 + RN^2 = CA^2 + CQ^2 = AQ^2.$$

19. Let  $AQ, AR$  be the fixed straight lines, and  $P$  the middle point of  $QOR$ .

Through  $C$  the middle point of  $AO$  draw  $CH, CK$  parallel to  $AQ, AR$ ,

and through  $P$  draw  $PHM, PKN$  parallel to  $AR, AQ$ .

Then the complements  $AC, HK$  about the diagonal  $MN$  are equal.

Therefore  $PH \cdot PK$  is constant, and  $P$  lies on a rectangular hyperbola, having  $CH$  and  $CK$  for asymptotes.

20. Draw  $QB$  perpendicular to  $AB$ , and make  $AB$  equal to  $CD$ , then  $A$  is a fixed point.

Then  $AD : AB :: BC : CD :: QB : DP$ ,

or  $PD \cdot DA = AB \cdot BQ$ .

Hence  $P$  lies on a rectangular hyperbola of which  $AB$  is one asymptote.

21. If  $D, E, F, O$  be the centres of the escribed and inscribed circles,

$$OC \cdot CF = DC \cdot CE,$$

since the triangles  $OCE, DCF$  are similar.

Hence the hyperbola is rectangular since diameters at right angles are equal.

22. If the diameters of the parallelogram  $LML'M'$  meet in  $C$ , the angle  $SLS' = SL'S'$ .

Hence  $S$  and  $S'$  lie on a rectangular hyperbola circumscribing  $LML'M'$ . (Art. 137.)

23. If  $PSq, SQq$  be the chords and  $D$  be a point on the directrix, such that  $DS$  bisects the angle  $QSp$ , then  $D$  will lie in  $pq$ .

But  $DS$  is perpendicular to the asymptote since  $Pp, Qq$  are equally inclined to it, therefore  $D$  lies on the asymptote.

Similarly  $Pq, Qp$  meet in  $D'$ , the foot of the perpendicular from  $S$  on the other asymptote.



24. If  $V$  be the middle point of  $PQ$  and  $POP'$  be a diameter,

the angle  $VNP = NPV = QP'P = POV$ .

Hence  $O, P, V, N$  lie on a circle.

If  $OQ$  meet the given tangent in  $T$ , produce  $OQ$  to  $V$ , making  $OV$  a third proportional to  $OT$  and  $OQ$ ; with centre  $V$  and radius, a mean proportional between  $VO$  and  $VT$ , describe a circle meeting the given tangent in  $P$  its point of contact. In the tangent measure off

$$PL = PM = OP,$$

then  $OL$  and  $OM$  are the asymptotes.

25. Draw  $PD$  perpendicular to the base  $QR$ ,

then, since  $PD^2 = DQ \cdot DR$ ,

$DP$  is the tangent at  $P$ .

26. Let a circle on  $DE$  meet the hyperbola in  $P$  and  $Q$ , draw the diameters  $PCP'$ ,  $QCQ'$ .

Then, since the angle  $DPE$  is either equal or supplementary to  $DP'E$  and  $DQE$  to  $DQ'E$ , the similar circle on the other side of  $DE$ , will meet the curve in  $P'$  and  $Q'$ .

27. Let  $OAD$ ,  $OBC$  be the fixed straight lines,  $PM$ ,  $PN$ ,  $PL$  perpendiculars from the centre of the circle on  $BC$ ,  $AD$  and the bisector of the angle  $AOB$ .

Let  $PM$ ,  $PN$  meet  $OL$  in  $m$ ,  $n$ .

Draw  $Ll$ ,  $Ll'$ ,  $Pr$ ,  $Pr'$  perpendicular to  $BC$ ,  $AD$ ,  $Ll$ ,  $Ll'$  respectively.

Then,  $BM^2 + MP^2 = AN^2 + NP^2$ ,

or  $PN^2 - PM^2 = BM^2 - AN^2$

which is constant,

$$PN^2 = (Ll' + Ll)^2 = (Ll - Ll')^2 + 4Ll \cdot Ll' = PM^2 + 4Ll \cdot Ll'.$$

Hence the rectangle  $Ll \cdot Ll'$  is constant;

but  $Ll : OL$  in a constant ratio and  $Ll' : PL$  is constant.

Therefore  $PL \cdot LO$  is constant, or  $P$  lies on a rectangular hyperbola having  $OL$  for an asymptote.

28. If  $P$  be the point of contact

$$CL = 2PN, CL' = 2CN;$$

therefore  $CL \cdot CL' = 2Ca^2 = 4PN \cdot CN$ .

Hence  $CN \cdot CL' : Ca^2 :: Ca^2 : PN \cdot CL$ ,

or  $AC^2 : Ca^2 :: Ca^2 : CB^2$ .

29. Draw  $CV$  conjugate to  $PQ$ .

Then, the angle

$$TPQ = PP'Q = PCV = CPQ';$$

therefore the angles  $CPQ, TPQ'$  are equal.

30. Let  $DB, DC$  meet the asymptotes in  $b$  and  $c$ : draw  $AH, AK$  parallel to the asymptotes. Then  $OBb, OAK$  are similar triangles, also  $OCc, OAK$ .

Hence

$OB : OA :: Ob : OK :: OH : Oc :: AK : Oc :: OA : OC$ ;  
therefore  $A$  lies on the circle of which  $BC$  is diameter as does  $D$ .

31. Let the tangents at  $P$  and  $Q$  meet the asymptote in  $L$  and  $M$ .

The angle

$$\begin{aligned} PCQ &= PCL - QCM = PLC - CMT = LTM \\ &= \text{supplement of } PTQ. \end{aligned}$$

32. Each hyperbola passes through the orthocentre of the triangle  $ABC$ .

Hence  $D$  is that orthocentre.

Now the line joining the middle point of  $AB$  to the middle point of  $CD$  is a diameter of the nine point circle.

And  $AB^2 + CD^2 = \text{square on diameter of circumscribed circle}$ .

Hence the circles intersect at right angles.

$$33. PN^2 = CN^2 - CA^2 = CN^2 - CN \cdot CT = CN \cdot NT.$$

Hence the triangle  $CPN$  is similar to  $PTN$ , and therefore to  $tTC$ .

34. This problem is a particular case of Ex. 61 on the hyperbola.

35.  $CM, CN$  which bisect  $PP', PQ'$  are conjugate being equally inclined to the asymptotes;  
therefore  $P'Q'$  is a diameter.

36. If  $AB$  be a diameter of the hyperbola,  $CD$  subtends, at  $A$  and  $B$ , angles which are both equal and supplementary, and are therefore right angles.

37. If  $AQ', BQ$  meet in  $R$ , the angle  $QAR =$  supplement of  $QBR = RBP$ ;  
therefore  $R$  lies on the circle.

38. Let  $CV, CV'$  bisect  $PQ, P'Q$ .  
The angle  $PRQ = PQL = VCQ = CQV'$ ,  
so  $P'R'Q = CQV$ .

Draw  $VM$  perpendicular to  $CQ$ ,  
then  $PQ : QR :: VM : CV$ ,  
and  $P'Q : R'Q :: VM : QV$ .  
Now  $P'Q = 2CV$ ,  
and  $PQ = 2QV$ ;  
therefore  $QR = QR'$ .

39. Let  $PN$  meet  $CF$  in  $K$ ,  
then  $CF$  varies as  $CG$ , and  $PF$  varies as  $PK$ , or  $Ct$ , or  $CT$ .

Hence  $PF \cdot FC$  is proportional to  $CG, CT$  or  $CS^2$ .

Hence  $P$  lies on a rectangular hyperbola having  $CF$  for asymptote.

40. If the tangent at  $Q$  meet in  $V$  the line joining the fixed points  $A$  and  $B$ ,  
 $VQ^2 = VA \cdot VB$ .

41. If the chord  $QR$  meet the tangent at  $P$  in  $E$ ,  
 $RPL = QPE = PRQ$ .

42. If  $D$  be the point,

$$CD \cdot CT = 2AC^2 \text{ and } CD = AC.$$

43.  $CP = CD$  and are equally inclined to the asymptote.

44. Let  $BAD$  be the given difference, and draw  $CL$  parallel to  $AD$ , meeting  $BA$  in  $L$ .  $CAL$ ,  $CBL$  are similar triangles;

$$CL^2 = AL \cdot BL.$$

45. If  $tQT$ ,  $t'QT'$  be the tangents, and  $SM$  perpendicular to the axis,

$$\begin{aligned} SQM &= SQT - MQT = S'Qt' - CtT \\ &= QS'M + QT'C - CtT = QS'M; \\ \therefore QM^2 &= SM \cdot S'M. \end{aligned}$$

46. If  $V$  is the middle point of  $OP$ ,

$$\begin{aligned} OTV &= VOT, \therefore OTP \text{ is a right angle,} \\ &= OCV, \text{ Art. 136;} \end{aligned}$$

$\therefore O, V, C, T$  are concyclic, and

$$\therefore VCT = VOT = OCV.$$

Similarly, if  $U$  is the middle point of  $OQ$ ,

$$OTU = UTQ.$$

## CHAPTER VI.

### THE CYLINDER AND THE CONE.

1. TAKE two points  $E$  and  $A$  on the generating line, and draw  $EX$  at right angles to the axis, making  $EX$  equal to  $EA$ .

Then the plane containing  $AX$ , and perpendicular to the plane  $EA X$  will cut the cylinder in an ellipse of the required eccentricity.

2. Take two points  $E$  and  $A$  on the generating line and with centre  $A$ , and radius twice  $EA$ , describe a circle meeting  $EF$  in  $X$ .

Then the plane through  $AX$  perpendicular to the plane  $EXA$  will intersect the cone in an ellipse of the required eccentricity.

3. Take two points  $EA$  on the generating line, the least angle of the cone will be, when  $EA$  is double the perpendicular from  $A$  on  $EF$ , that is when the semi-vertical angle is equal to the angle of an equilateral triangle.

4. A tangent plane to a cone touches it along a generating line  $OF$ , hence  $OF$  is parallel to all sections parallel to the tangent plane which are therefore parabolas.

If  $C$  be the centre of the sphere  $FES$ , the ratios  $CS : CA$  and  $CA : CO$  are constant, and the angle  $OCS$  is constant, therefore  $COS$  is constant, and  $S$  lies on a cone of which  $O$  is vertex and  $OC$  axis.

5. Through the flames of the candles which are treated as points, draw planes intersecting in the ceiling, in a straight line, since these fixed planes must always be tangent planes to the ball, the locus of the centre of the ball is a horizontal straight line.

6. The triangles  $AEX$ ,  $A'E'X'$  have all their sides parallel;

therefore

$$SA : AX :: EA : AX :: E'A' : A'X' :: S'A' : A'X'.$$

7. If  $C$  be the centre of the sphere  $FES$ , the angle  $OCS$  is constant.

And the ratios  $CE : CA$ ,  $CE : CV$ , and  $CS : CA$  are all constant ;

therefore  $CS : CV$  is constant and the angle  $CVS$  is constant, or  $VS$  is a fixed straight line.

8. Take two points  $E$  and  $A$  on the generating line and with centre  $A$  and radius  $AX$ , such that  $EA : AX$  in the ratio of the eccentricity, describe a circle intersecting  $FE$  in  $X$ , the section of which  $AX$  is axis will have the required eccentricity.

9.  $XS$ ,  $XS'$  are tangents to the same sphere  $FES$  of which  $C$  is centre.

Let  $SS'$ ,  $EF$  meet the axis in  $V'$ ,  $L$ , and let  $CX$  meet  $SS'$  in  $M$ .

Then 
$$CL \cdot CV' = CM \cdot CX = CE^2.$$

Hence  $V$  and  $V'$  coincide.

10. Draw  $CN$  perpendicular to the axis,

then 
$$2CN = A'D' - AD,$$

and 
$$2ON = OD + OD'.$$

In  $CN$  take a point  $Q$ , such that

$$QN : CN :: DO^2 : DA^2;$$



then since  $OD' - OD : 2CN :: OD : DA,$   
 $2QN : OD' - OD :: DO : DA;$

also  $AD + A'D' : 2ON :: AD : DO,$

and  $AA'^2 = DD'^2 + (AD + A'D')^2.$

Hence  $QO^2 : CA^2 :: OD^2 : DA^2.$

Hence  $Q$  lies on a sphere of which  $O$  is centre, and therefore  $C$  lies on a spheroid, having  $OD$  for its axis, which is oblate or prolate according as  $DO$  is greater or less than  $DA$ , that is according as the vertical angle of the cone is greater or less than a right angle.

11. Draw a plane  $CT$  through  $C$ , the centre of the sphere perpendicular to the axis intersecting the tangent plane at  $S$  in  $T$ . Then since the angles  $COE$  and  $CTS$  are equal, and  $CE = CS$ , it follows that  $CT = CO$ .

Hence  $S$  lies on the surface generated by the revolution of the circle of which  $CT$  is diameter.

12. Only two circles can be described passing through  $S$ , and touching the generating lines in which the plane through  $S$  and the axis intersects the cone.

If  $ST, ST'$  be the tangents, and  $OS$  meet the circles in  $D$  and  $D'$ ,

the angle  $DST = \text{half } SCD = \text{half } SC'D' = D'ST'.$

Hence the planes of the corresponding sections make equal angles with  $OS$ .

13. If the plane of section meet the plane through  $O$  perpendicular to the axis in the line  $ZZ'$ , and  $PK$  be perpendicular to  $ZZ'$

$OP$  or  $OQ$  bears to  $PK$  a constant ratio.

Hence if  $P'$  in the projection corresponds to  $P$

$OP : P'K$  in a constant ratio.

But  $OP'$  is equal to the perpendicular from  $P$  on the axis which bears to  $OP$  a constant ratio.

Hence the ratio  $OP' : P'K$  is constant.

14.  $A'Q - QA = SS'$  and a right cone can be constructed of which  $Q$  is vertex, such that the generating lines intersect the ellipse.

Hence

$$PQ + AS = AS + QR + RP = AE + EQ + SP = AQ + SP.$$

15. Through the vertical straight line which is the locus of the luminous point draw vertical tangent planes to the ball intersecting the inclined plane in  $OY, OZ$ . The locus of  $C$  the centre of the shadow will be the straight line bisecting  $YZ$ , at right angles,  $SY, SZ$  being perpendiculars from the point of contact of the ball on  $OY, OZ$  which are tangents to the elliptic shadow.

16. The given plane to which the sections are perpendicular must be supposed to contain the axis of the cone.

Take  $OK$  on the generating line equal to  $AS$ .

Draw  $KG$  perpendicular to  $OK$ , meeting a line through  $O$  at right angles to the axis in  $G$ , then  $G$  is a fixed point.

Draw  $GM$  perpendicular to  $AS$ ,

then  $CE : EO :: OK : KG$ ,

or  $KG : KA :: AE : CE$ .

Hence the angle  $KAG = ACE = \text{half } KAM$ .

Hence  $AS$  touches a circle centre  $G$  and radius  $GK$ .

$$17. VP = VQ = VA + AQ = 2AE + AN = 2AS + AN.$$

$$18. EX', X'A' \text{ will be parallel to } EX, XA.$$

Hence if  $AF$  be drawn parallel to  $A'E'$ , the ratio of the eccentricities is  $AE : AF$  which is constant.

19. The volume varies as the area  $AVA'$  and  $BC$ ; and the area  $AVA'$  varies as  $AV \cdot VA'$ , or  $AD \cdot A'D'$ , that is,  $BC^2$ .

Hence  $BC$  is constant.

$$20. \quad A'O - OA = A'E' - EA = A'S - SA = SS'.$$

Hence the locus of  $O$  is an hyperbola of which  $A$  and  $A'$  are foci.

$$21. \quad BC^2 = EC \cdot CE'.$$

Hence the locus of  $C$  is an hyperboloid of revolution generated by the revolution round the axis of the cone of an hyperbola of which  $OE, OE'$  are asymptotes.

$$22. \quad CA \text{ and } EA \text{ are the bisectors of the angle } OAS.$$

Hence the sphere on  $CE$  as diameter, intersects the plane of section in a circle of which  $AA'$  is diameter.

## CHAPTER VII.

### CURVATURE.

1.  $SL$  being a fourth of the chord of curvature at  $L$  through  $S$ ,  $LG$  the normal is a fourth of the diameter of curvature.

2. The normal and tangent at  $L$  are equally inclined to the axis.

3. Draw  $SO$  at right angles to  $SP$ , meeting the normal at  $P$  in  $O$ , and from  $O$  draw  $OQ$  at right angles to  $AP$ ; then the chord of curvature through  $A$  is  $4PQ$ ; drawing  $AZ$  perpendicular to  $TP$ ,

$$PQ : OP :: AZ : AP,$$

or

$$PQ \cdot AP = OP \cdot AZ.$$

Again,

$$SP : PO :: AZ : AT,$$

or

$$OP \cdot AZ = SP \cdot AT;$$

also

$$PY : SP :: AT : TY,$$

or

$$PY^2 = SP \cdot AT = PQ \cdot AP.$$

Hence

$$4PQ : 4PY :: PY : AP.$$

4. The diameter at  $P$  and  $SP$  are equally inclined to the normal at  $P$ ; hence chord of curvature parallel to the axis is equal to  $4SP$ .

5. Let  $QM$  be the ordinate of  $Q$ .

Then  $QM : MG :: PN : NG$ ,

or  $PN.MG = 2AS.QM = pn.Mg$ .

Also  $Nn = Gg$ .

Hence  $MG : Mg :: pn : PN$ ,

or  $MG : Gg :: pn : pn - PN$ .

Therefore  $MG : pn :: Nn : pn - PN :: pn^2 - PN^2$   
 $: 4AS(pn - PN) :: pn + PN : 4AS$ .

But  $QM : MG :: PN : 2AS$ ;

hence  $QM : 2PN :: pn(pn + PN) : 16AS^2$ .

If  $P$  and  $p$  approach each other indefinitely,  $Q$  is the centre of curvature,

and  $PQ : PN + QM :: PG : PN$ .

But  $QM : PN :: PN^2 : 4AS^2 :: AN : AS$ ;

hence  $QM + PN : PN :: AN + AS : AS :: SP : AS$ .

Hence  $PQ : PG :: SP : AS :: PG^2 : SR^2$ .

6. If  $PK$  be perpendicular to the directrix and  $S'$  the farther focus,

$$SP : PK :: CA : CX;$$

hence  $SP = \frac{1}{2}AC$ ,

and  $S'P = \frac{3}{2}AC$ .

Therefore  $PE.PS' = \frac{3}{2}AC^2 = 2SP.S'P = 2CD^2$ ,

or the circle of curvature passes through  $S'$ .

7. If  $PV$  be the chord of curvature through the focus and  $pp'$  the focal chord parallel to the tangent at  $P$ ,

$$PV.AC = 2CD^2,$$

or  $PV.AA' = DD^2 = AA'.pp'$ ;

hence  $PV = pp'$ .

8. Let  $Q$  be the middle point of the chord and  $QM$  its ordinate.

If  $PNP'$  be the double ordinate,  $P'Q$  is a diameter.

If  $PQ$  meet the axis in  $W$ ,

$$WM = TN = NW ;$$

hence

$$AM = 5AN,$$

and

$$QM^2 = PN^2 = 4AS \cdot AN = \frac{4}{5}AS \cdot AM ;$$

or  $Q$  lies on a parabola of which  $A$  is vertex and  $AS$  axis.

Produce  $SA$  to  $S'$  making  $S'A = 3AS$ , and let  $PQ$  meet the tangent at  $A$  in  $Y'$ .

Then  $AY'^2 = \frac{9}{4}PN^2 = 9AS \cdot AN = 3S'A \cdot AN = S'A \cdot AW$ .

Hence  $S'Y'$  is perpendicular to  $PQ$ , and  $PQ$  envelopes a parabola of which  $S'$  is focus and  $A$  vertex.

9.  $DR$  and  $PCP'$  are equally inclined to the axis, and  $D, R, P, P'$  lie on a circle ;

hence  $PR, DP'$  are equally inclined to the axis.

So  $DQ, PD'$  are equally inclined to the axis ;

hence  $PR, DQ$  are parallel, since  $DP', PD'$  are parallel.

$$10. \quad PG = CD = CP.$$

Hence the radius of curvature varies as  $CP^3$ .

$$11. \quad LP \cdot PC = PT \cdot Pt = CD^2.$$

Hence  $PL$  is equal to half the chord of curvature in direction  $PC$ .

$$CP \cdot CL = CP \cdot PL + CP^2 = CD^2 + CP^2 = AC^2 + BC^2.$$

12. The circle on  $PE$  as diameter touches the curve at  $P$  and goes through  $Q$  ; when  $Q$  coincides with  $P$  the circle becomes the circle of curvature.

13. If the tangents at two near points  $P$  and  $Q$  meet in  $T$ ,

$$TP : TQ :: CD : CE.$$



Hence the difference of  $TP$  and  $TQ$  is very small compared with either, and if a circle be described, of which the intersection of normals at  $P$  and  $Q$  is centre, to touch  $TP$  and  $TQ$  at  $P$  and  $Q$ , when  $P$  and  $Q$  coincide this becomes the circle of curvature at  $P$ .

14. If  $C$  be the centre of curvature at the vertex,

$$AC = 2AS.$$

If  $PR$  be the tangent,

$$\begin{aligned} PR^2 &= CP^2 - CR^2 = PN^2 + CN^2 - 4AS^2 \\ &= 2AC \cdot AN + CN^2 - AC^2 = AN^2. \end{aligned}$$

15.  $PCP'$  and the tangent at  $P$  are equally inclined to the axis.

16. If  $O$  be the centre of curvature,

$$PO^2 = AC \cdot C = PF \cdot CD.$$

$$\text{But} \quad PO \cdot PF = CD^2,$$

$$\text{hence} \quad PO = CD = PF.$$

Hence, if with centre  $C$  and radius  $CP$ , such that

$$CP^2 = AC^2 + BC^2 - AC \cdot BC,$$

a circle be described, it will meet the curve in points, at which the radius of curvature has the required value.

17. If  $PV$  be the chord and  $CQ$  be drawn parallel to  $Pt$ ,

$$2CD^2 = PQ \cdot PV = Ct \cdot PV.$$

$$\text{But} \quad Ct \cdot PM = BC^2,$$

$$\text{hence} \quad PV : PM :: 2CD^2 : BC^2.$$

18. If  $PQ$  be the common chord of the ellipse and circle of curvature,  $TPt$  must make equal angles with both axes, or  $CT = Ct$ .

Make the angle  $ACP$  such that

$$PN : NC :: BC^2 : AC^2;$$

then  $CP$  will meet the ellipse in the point required.

19.  $SP$  is one-fourth of the chord of curvature through  $S$ , hence  $PQ$  is half the radius of curvature.

20. If  $PV$  be the chord,

$$PV \cdot PD = 2CD^2.$$

But

$$PN = ND,$$

hence

$$PV : CD :: CD : PN.$$

$$21. \quad PG : PF :: PK : PC :: PE : Pg,$$

or

$$PF \cdot PE = PG \cdot Pg = CD^2;$$

hence  $E$  is the centre of curvature.

22. Draw  $SQ$  at right angles to  $SP$  to meet the normal in  $Q$ ; let the tangent at  $P$  meet the directrix in  $Z$ .

$$\text{Then} \quad PL : PS :: ZP : ZS \therefore PQ : PS.$$

Therefore  $PL = PQ$  = half the radius of curvature.

23. If  $CE$  bisect  $PQ$ ,  $PCE$  is a right angle and

$$PF \cdot PE = CP^2 = CD^2;$$

hence  $PQ$  is equal to the diameter of curvature at  $P$ .

$$24. \quad OP : CP :: PQ : PF,$$

or

$$OP \cdot PF = CP^2 = CD^2;$$

hence  $O$  is the centre of curvature at  $P$ .

25. If the normal at  $P$  meet  $BC$  in  $K$ ;  $S, P, H, K$  lie on a circle; hence, when  $P$  coincides with  $B$ ,  $K$  becomes the centre of curvature at  $B$ .

26. If  $HT$  be produced to  $H'$  making  $H'T = TH$ ,  $TQ, PH'$  will be parallel;

$$\text{hence} \quad PR : RS :: H'T : TS :: TH : TS.$$

$$\text{Therefore} \quad PR : PS :: TH : 2TC :: HP : 2CY :: HP : 2CA,$$

or

$$2PR \cdot AC = PS \cdot HP = CD^2.$$

Hence  $PR$  is one-fourth of the chord of curvature through  $S$ .

27. If  $O$  be the centre of curvature at  $A$ ,

$$SO : SA :: SA : AX.$$

Hence curvature of ellipse is greater than that of parabola, and curvature of parabola is greater than that of hyperbola.

28. By Ex. 6,  $SP = \frac{3}{2}AC$ ,

and  $SP : CY' :: PE : PC.$

Hence  $PE : EP' :: 3 : 1.$

29. Let  $CQ'$  be conjugate to  $CF$  and  $CP'$  parallel to  $PQ$ .

Then  $OE^2 : OF^2 :: OQ.OP : OF^2 :: CP'^2 : CQ'^2$   
 $:: CD^2 : CQ'^2 :: TP^2 : TF^2 :: TE^2 : TF^2.$

Hence  $TEOF$  is cut harmonically.

30. Project the angle between the common diameters into a right angle; the ellipses obtained will be inscribed, symmetrically, in a square, and will therefore be equal.

$$\begin{aligned} 31. \quad SE^2 &= Py^2 + (EP - Sy)^2 \\ &= SP^2 + EP^2 - 2EP.Sy \\ &= EP^2 - 3.SP^2. \end{aligned} \quad \text{Art. 160.}$$

32. Let  $F$  be the middle point of  $PE$ , the radius of curvature at  $P$ .

Then  $PF.SY = SP^2$ , Art. 160.

$$\therefore SY : SP :: SP : PF,$$

$\therefore SYP, SPF$  are similar triangles, and the angle  $PSF$  is a right angle, so that the locus of  $S$  is the circle, diameter  $PF$ .

## CHAPTER VIII.

### PROJECTIONS.

1. THE theorems are obtained by projection from the following properties of the circle.

Art. (65) Every diameter of a circle is bisected at the centre, and the tangents at its extremities are parallel.

(70) If the chord of contact of tangents from  $T$  to a circle meet  $CT$  in  $N$ ,  $CT \cdot CN = CA^2$ .

(71) If  $M, t$  correspond to  $M', t'$  on the circle,

$$CM : CM' :: BC : AC :: Ct : Ct' ;$$

therefore

$$CM \cdot Ct = BC^2.$$

(73) If the chord of contact of tangents from  $T$  to a circle meet  $CT$  in  $V$  and  $CT$  meet the curve in  $P$ ,

$$CT \cdot CV = CP^2.$$

(74) A diameter of a circle bisects all chords parallel to the tangents at its extremities.

(75) If a diameter of a circle bisects chords parallel to a second, the second diameter bisects all chords parallel to the first.

(76) Art. 77 is meant.

If  $PCP', DCD'$  be diameters of a circle at right angles,

$$QV^2 : PV \cdot VP' :: CD^2 : CP^2.$$

(78) If  $CA, CB$  be radii at right angles,

$$CN^2 = AM \cdot MA', \quad CM^2 = AN \cdot NA',$$

$$CM : PN :: AC : BC :: CN : DM.$$

(81) The area of a square circumscribing a circle is constant and equal to the rectangle contained by diameters at right angles.

(82) If  $CD, CP$  be radii at right angles and the tangent at  $P$  meet a pair of radii at right angles in  $T$  and  $t$ ,

$$PT \cdot Pt = CD^2.$$

(83) Cor. 1. The two tangents  $TP, TQ$  from any point are equal, and the parallel diameters  $ACA', BCB'$  are equal,

$$\therefore TP : ACA' :: TQ : BCB';$$

and these ratios are unaltered by projection.

Cor. 2. In the circle  $TCT'$  is a right angle,

$$\text{and} \quad \therefore PT \cdot PT' = CP^2 = CD^2,$$

$CD$  being parallel to  $TPT'$ .

(84) Taking  $ABA'$  as a semicircle,  $ECF$  is a right angle; project on any plane parallel to the line  $ACA'$ .

2. If a parallelogram be inscribed in a circle its sides are at right angles. The greatest rectangle than can be inscribed in a circle is a square having its area equal to  $2AC^2$ ; hence the greatest parallelogram that can be inscribed in an ellipse has its area equal to  $2AC \cdot BC$ .

3. The theorem is true in the case of a circle, and follows by projection.

4. The greatest triangle which can be inscribed in a circle is an equilateral triangle of which  $C$  is the centre of gravity.

Produce  $PC$  to  $V$ , making  $2CV = PC$ ;

then, if  $QVQ'$  be the ordinate,  $PQQ'$  is the greatest triangle which can be inscribed in the ellipse having its vertex at  $P$ .

5. If a straight line meet two concentric circles, the portions intercepted between the curves are equal.

6. The locus of the point of intersection of tangents at the extremities of diameters of a circle at right angles is a concentric circle.

7. The locus of the middle points of lines joining the extremities of diameters of a circle at right angles is a concentric circle.

8. If  $CP$ ,  $CD$  be radii of a circle at right angles and  $CA$  bisect the angle  $PCD$ , the tangent at  $A$  meets  $CP$  in  $T$  such that  $PD^2 = 2AT^2$ .

9. If a chord  $AQ$  of a circle be produced to meet the diameter at right angles to  $CA$  in  $O$  and  $CP$  be parallel to  $AQ$ ,

$$AQ \cdot AO = 2CP^2.$$

10. If  $OQ$ ,  $OQ'$  are tangents to a circle and  $R$  be a diagonal of the parallelogram of which  $OQ$ ,  $OQ'$  are adjacent sides, then if  $R$  be on the circle the locus of  $O$  is a concentric circle.

11. If a parallelogram be inscribed in a circle and from any point on the circle straight lines are drawn parallel to the sides of the parallelogram, the rectangles under the segments of these lines made by the sides are equal to one another.

12. If a square circumscribe a circle and a second square be formed by joining the points where its diagonals meet the circle, the area of the inner square is half that of the outer. And if four circles be inscribed in the spaces between the outer square and the circle, their centres will lie on a concentric circle.

13. If a rectangle be inscribed in a circle so that the diameter bisecting one pair of sides is divided in a constant ratio, the area is constant.

14. If a parallelogram circumscribe a circle and one of its diagonals bear a constant ratio to the diameter it contains, the area is constant.



15.  $PQR$  is a triangle inscribed in a circle, the centre being the intersection of lines joining the angular points to the middle points of opposite sides. If  $PC$ ,  $QC$ ,  $RC$  meet the circle again in  $P'$ ,  $Q'$ ,  $R'$ , the tangents at  $P'$ ,  $Q'$ ,  $R'$ , will form a triangle similar to  $PQR$ , its area being four times as great.

16. The locus of the middle points of chords of a circle passing through a fixed point is a circle of which the line joining that point to the centre is diameter.

17. The ellipse which touches the middle points of the sides of a square (i.e. a circle) is greater than any other inscribed ellipse.

18. If a polygon circumscribe a circle, its area is a minimum when any side is parallel to the line joining the points of contact of adjacent sides.

19. The greatest triangle which can be inscribed in a circle has one side bisected by a diameter and the others cut in points of trisection by the diameter at right angles.

20.  $AB$  is a given chord of a circle,  $C$  any point of the circle, the locus of the intersection of the straight lines joining  $A$ ,  $B$ ,  $C$  to the middle points of  $BC$ ,  $CA$ ,  $AB$  is a circle.

21. If  $CP$ ,  $CD$  are radii of a circle at right angles, the circle on  $PD$  as diameter will go through  $C$ .

22. The theorem is true in the case of a circle intersecting a concentric rectangular hyperbola, and follows generally by projection.

23. If  $V$  is the middle point of  $Qq$ , project  $CVQ$  into a right angle.

24. If  $PT$ ,  $pt$  are tangents at the extremities of a diameter  $Pp$  of a circle, then if any diameter meet  $PT$  in  $T$  and the diameter at right angles meet  $pt$  in  $t$ , and any tangent meet  $PT$  in  $T'$  and  $pt$  in  $t'$ ,

$$PT : PT' :: pt' : pt.$$

25. If  $CP$ ,  $CD$  be radii of a circle at right angles and  $Pp$ ,  $Dd$  be drawn parallel to any tangent, and any line through  $C$  meet  $Pp$ ,  $Dd$  and the tangent in  $p$ ,  $d$  and  $t$ ,

$$Cp^2 + Cd^2 = Ct^2.$$

26. If  $ACA'$ ,  $BC$  and  $CD$ ,  $CP$  be pairs of radii of a circle at right angles and if  $BP$ ,  $BD$  be joined, also  $AD$ ,  $A'P$ , the latter intersecting in  $O$ ,  $BDOP$  is a parallelogram.

27. If  $TM$  be perpendicular to  $SP$ ,

$$TM : TP :: SY : SP :: BC : CD;$$

hence

$$TP : CD \text{ is constant.}$$

If a point be taken on a tangent to a circle such that its distance from the point of contact is constant and therefore proportional to the parallel radius, its locus is a concentric circle.

## CHAPTER IX.

### CONICS IN GENERAL.

1.  $TN : XN :: SR : SX :: SP : XN;$

hence  $SP = TN.$

Also  $TP \cdot TP' = TN^2 - PN^2 = SP^2 - PN^2 = SN^2.$

2. Draw  $Pm, Qn$  perpendicular to the directrix.

Then  $PR : QN :: KP : KQ :: Pm : Qn :: SP : SQ :: PM : QN,$

or  $PR = PM.$

3.  $PS \cdot SQ : AS \cdot SA' :: Cp^2 : CA^2,$

and  $PQ \cdot SR = 2SP \cdot SQ.$

Hence  $PQ$  varies as  $Cp^2.$

4. Let  $Pp, Qq$  intersect in  $O.$

Then  $QO \cdot Oq : PO \cdot Op :: TP^2 : TQ^2 :: QO^2 : PO^2.$

Hence  $TO$  bisects  $pq$  as well as  $PQ$ , and is a diameter and goes through  $t.$

5. Since  $RS$  is the exterior bisector of the angle  $P'SQ',$

$$SP' : SQ' :: RP' : RQ'.$$

6. Let  $S'T$ ,  $ST$  meet  $PQ$  in  $EE'$ , and let  $TF$ ,  $TG$  be the common interior and exterior bisectors of the angles  $ETE'$ ,  $PTQ$ .

Bisect  $FP$  in  $O$ .

Then  $OE \cdot OE' = OF^2 = OP \cdot OQ$ .

Now  $RP : RQ :: PE : EQ$ ,

and  $R'P : R'Q :: PE' : E'Q$ .

Again,  $OP : OE :: OE' : OQ$ ;

hence  $PE : OE :: E'Q : OQ$ .

Also  $OP : OE' :: OE : OQ$ ;

hence  $PE' : OE' :: EQ : OQ$ .

Therefore  $RP \cdot PR' : RQ \cdot QR' :: PE \cdot PE' : EQ \cdot QE'$   
 $:: OE \cdot OE' : OQ^2 :: OF^2 : OQ^2$ .

Hence  $TF$  bisects the angle  $RTR'$ , and the angles  $RTP$ ,  $R'TQ$  are equal.

7.  $TS$  and  $KS$  are the interior and exterior bisectors of the angle  $PSP'$ .

Hence if  $ST$  meet  $PP'$  in  $E$ ,

$RK : TE :: KP : EP :: KP' : EP' :: KR' : ET$ ,

or  $RK = KR'$ .

8. If  $DE$ ,  $D'E'$  are perpendicular to  $SP$ ,  $SP'$ ,  
 then  $SE = SE'$ .

Hence  $DE$ ,  $D'E'$  intersect on the bisector of the angle  $PSP'$ , which is  $ST$ .

9. If  $PP'$  meet the directrix in  $K$ ,  $PP'$  is harmonically divided at  $S$  and  $K$ .

Hence any chord through  $S$  is harmonically divided by the directrix and the tangents at  $P$  and  $P'$ .

10. Draw  $SY$  perpendicular to the tangent, then since  
 $SC : CY :: SA : AX$ ,  
 the locus of  $C$  is a circle.

11. Let  $Pp$  meet the curve in  $P'$ , and let  $QP'$  meet the directrix in  $q'$ .

Then since  $pS, q'S$  are the exterior bisectors of the angles  $PSP', QSP'$ , the angle  $pSq' = \text{half } PSQ = pSq$ ; hence  $q$  and  $q'$  coincide.

$$12. \quad SL : SP :: FT : FP :: TN : PK.$$

Therefore  $SL : TN :: SP : PK :: SA : AX.$

$$13. \quad PN^2 : AC^2 - CN^2 :: CB^2 : CA^2 :: Cb^2 : Ca^2 \\ :: pn^2 : Cn^2 - Ca^2.$$

Hence, if  $PN = pn, AC^2 - CN^2 = Cn^2 - Ca^2,$

or  $CN^2 + Cn^2 = CA^2 + Ca^2.$

$$14. \quad QR : LG :: PQ : PG :: PM : PN.$$

Hence  $QR : PM :: LG : PN :: SG : SP :: SA : AX.$

15. Draw  $KV$  parallel to the axis.

Then  $VK : VP :: GS : SP :: SG' : SQ :: VK : VQ,$

or  $PV = VQ.$

Also  $PL : PM :: PK : PG :: PV : PS;$

hence  $2PL \cdot PS = PM \cdot PQ = SR \cdot PQ = 2SP \cdot SQ,$

or  $PL = SQ.$

Hence  $SV = VL$ , and the diagonal of the parallelogram  $SL$  goes through  $V$ .

16. If  $PQR$  be the triangle and  $S$  the focus, make the angles  $QSr, QSp$  each equal to the supplement of  $PSR$ .

Then, if  $PSq = PSr, p, q$  and  $r$  are the points of contact.

17. By Ex. 20, Chap. I., if  $KV$  be drawn parallel to the axis,  $PV = VQ.$

Hence  $PN : PL :: PK : PG :: PV : PS.$

Hence  $2PN \cdot PS = SR \cdot PQ = 2SP \cdot SQ,$

or  $PN = SQ.$

Hence  $SN = 2SV$ , and the locus of  $N$  is a similar conic.

18. Let  $CT, CT'$  meet the curve in  $p, d$ .

Then  $CT \cdot PR = 2Cp^2$ ,

and  $CT' \cdot QR = 2Cd^2$ .

Hence the triangle

$$CTT' : 2 \text{ triangle } Cpd :: 2Cpd : PRQ,$$

or the triangle

$$CTT' : AC \cdot BC :: AC \cdot BC : \text{the triangle } PRQ.$$

19. Let  $S$  be the centre of the circumscribed circle,  $H$  the ortho-centre,

then the feet of the perpendiculars from  $S$  and  $H$  on  $AB, BC, CA$  lie on the nine-point circle,

and the angle  $SAB = \text{complement of } C = HAC$ .

Therefore with  $S$  and  $H$  as foci a conic can be inscribed in  $ABC$ .

20. Let  $SV$  meet the directrix in  $Q$  and  $PK$  in  $Z$ , let  $QP$  meet the axis in  $C$ .

Then  $PZ : SP :: SG : SP :: SA : AX$ ;

hence  $PZ : PK :: SA^2 : AX^2$ .

Now  $SC : CX :: PZ : PK :: SA^2 : AX^2$ ,

or  $C$  is the centre.

$$21. DE \cdot DF : DC^2 :: AB^2 : AC^2 :: DG^2 : DC^2,$$

$$\text{or } DG^2 = DE \cdot DF.$$

22. By Ex. 74 on the parabola, if  $PGQ$  be the chord of contact,

$$DF : FG :: PG : GQ :: GF : FE,$$

$$\text{or } FG^2 = FD \cdot FE.$$

23. If  $Epg$  be drawn parallel to  $DTP$ ,

$$DP^2 : Ep \cdot Eq :: DF : EF.$$



For  $DF$  is parallel to a generating line  $VM$  of the cone of which the hyperbola is a section.

Draw  $Dlm$ ,  $LEM$  in the plane  $VFEM$  to meet the cone.

Then the sections of the cone by the parallel planes,  $lPm$ ,  $pLQ$  are similar,

and  $Dm = EM$ .

Hence

$$DP^2 : Ep \cdot Eq :: Dl \cdot Dm : EL \cdot EM :: Dl : EL :: DF : EF.$$

Again,  $Ep \cdot Eq : EQ^2 :: PT^2 : TQ^2 :: EK^2 : EQ^2$ ;

hence  $EK^2 = Ep \cdot Eq$  if  $Epq$  meet  $PQ$  in  $K$ .

And  $DG^2 : GE^2 :: DP^2 : EK^2 :: DP^2 : Ep \cdot Eq :: DF : EF$ .

Therefore  $FG^2 = FD \cdot FE$ .

24. Let the tangents at  $P$ ,  $Q$  and  $R$  meet  $EB$  in  $p$ ,  $q$  and  $r$ .

Then  $Ep \cdot Eq = EF^2$ , if  $PQ$  meet  $EB$  in  $F$ ;

also  $EB^2 = Er \cdot Ep$ ,  $EC^2 = Er \cdot Eq$ .

Hence  $EB^2 : EC^2 :: Ep : Eq$ , a constant ratio.

By Ex. 23. the same proposition is true in the case of an hyperbola if  $EB$  be parallel to an asymptote.

25.  $GK$  being perpendicular to  $SP$ ,

$$Pk : PK :: Pg : PG :: AC^2 : BC^2;$$

$\therefore Pk$  is constant.

Also  $kL : Pk :: SG : SP$ ;  $\therefore kL$  is constant.

26. Let the fixed line meet the curve in  $P$  and  $Q$ , and let the tangent at  $P$  meet  $SL$  in  $D$ , and the directrix in  $F$ ; then, Art. 11,  $SD : SF :: SL : SX$ .

The angle  $FSP$  is a right angle, so that  $SF$  is a fixed line, and,  $SX$  being a fixed line, the ratio of  $SF$  to  $SX$  is constant;  $\therefore SD$  is constant and  $D$  is fixed.

The envelope of  $PG$  is therefore the parabola, of which  $D$  is the focus and  $PQ$  the tangent at the vertex.

## CHAPTER X.

### HARMONICS, POLES AND POLARS.

1. If  $AOB$  be the common chord and  $PQOpq$  any transversal,

$$PO \cdot Op = AO \cdot OB = QO \cdot Oq.$$

2.  $OA$  is perpendicular to  $B'C'$  and meets it in  $D$ ,  
 $OA \cdot OD = \text{square on radius of concentric circle} = OB \cdot OE$   
 $= OC \cdot OF.$

Therefore  $OD = OE = OF.$

3. Draw  $ABC$  to be bisected by  $OB$  in  $B$ ,  $BEG$  to  $AO$  produced to be bisected by  $OC$  in  $E$ , and  $BFK$  to  $CO$  produced to be bisected by  $OA$  in  $F$ .

Then any one of the straight lines drawn through  $O$  parallel to  $AC$ ,  $BG$ ,  $BK$  will form a harmonic pencil with  $OA$ ,  $OB$ ,  $OC$ .

Draw  $BL$ ,  $BM$  parallel to  $OA$ ,  $OC$  to meet  $OC$ ,  $OA$  in  $L$  and  $M$ .

Then since  $OE = EL$ ,  $OF = FM$ ,  $EF$  is parallel to  $LM$  and is therefore bisected by  $OB$  and is also parallel to  $ABC$ .

Hence the pencil  $BC$ ,  $BE$ ,  $BO$ ,  $BF$  is harmonic.

4. Let the circles meet in  $P$ , bisect  $AC$  in  $E$ .

Then  $EB \cdot ED = EC^2 = EP^2$ ,

hence the circles cut at right angles.

5. By Art 182,  $PQ$ ,  $AE$ ,  $BD$  intersect in  $A$ , and  $A$   $\{B, E, Q, F\}$  is harmonic.

6.  $AP$ ,  $BQ$  and  $PB$ ,  $AQ$  meet each pair on the polar of  $O$  on which  $C$  lies ;

And the pencil formed by  $CB$ ,  $CO$ ,  $CA$  and the polar of  $O$  is harmonic.

7. If  $A$ ,  $B$  be points of contact and the third conic meet  $AB$  in  $C$  and  $D$ ,  $A$  and  $B$  are the foci of the involution,  $P$ ,  $Q$  are conjugate points.

Hence  $ACBD$  is a harmonic range.

8. Let the common chords meet in  $E$ , and let  $EPRQ$  be a tangent at  $R$  ; then since the common chords are one conic of the system,  $E$  and  $R$  are foci of the involution  $EPRQ$  and  $EPRQ$  is a harmonic range.

9. Let  $TP$ ,  $TQ$  be the tangents,  $TE$  any line, then  $F$  the pole of  $TE$  lies on  $PQ$  and  $PEQF$  is a harmonic range.

10. If the tangent at  $P$  meet the asymptotes in  $L$  and  $L'$ ,  $PL = PL'$ .

Hence the pencil  $CD$ ,  $CL$ ,  $CP$ ,  $CL'$  is harmonic.

11. A diameter is bisected at the centre : and the polars of the extremities of a diameter intersect at infinity.

12. If  $T$  be the pole of  $QR$  and  $H$  the second focus of the conic which touches the ellipse at  $Q$ ,

$$PQ + QH = SQ + QS' ;$$

or  $HS' = SP$ .

Therefore  $HS' + S'P = S'P + SP ;$

or  $S'$  is on the conic of which  $H$  is focus.

Again the angles  $TS'R$ ,  $TS'Q$  are equal, and  $PS'$ ,  $S'H$  are equally inclined to  $TS'$  ; hence  $TS'$  is a tangent at  $S$ .

Therefore  $T$  lies on the directrix of the conic of which  $P$ ,  $H$  are foci.

13. Let  $ST, S'T$  meet  $PQ$  in  $E, E'$ ; then the angles  $ETP, E'TQ$  are equal.

Now the ranges  $RPEQ, R'QEP$  are harmonic and  $QTP$  is common to the two pencils; hence the angles  $R'TQ, RTP$  are equal.

14. The middle points of all chords of the cone parallel to the given line, lie in a plane through the vertex, let this plane meet the given line in  $P$  and any section through it in  $A$  and  $A'$ .

Then  $Q$  the pole of the given line lies in  $AA'P$  and  $AQA'P$  is a harmonic range. Since  $VA, VA'$  are fixed generating lines,  $VQ$  is a fixed straight line.

15. If  $Tpq$  be the chord,  $P$  its pole, then  $PN$  the ordinate of  $P$  is the polar of  $T$ .

Let  $CP$  meet  $pq$  in  $v$  and the curve in  $Q$ .

Let  $QM, QG'$  be the ordinate and normal at  $Q$ .

If  $PG$  be drawn perpendicular to  $pq$ , it is parallel to  $QG'$ ;

hence  $CG : CG' :: CP : CQ :: CN : CM$ ;

Therefore  $CG : CN :: CG' : CM :: SC^2 : AC^2$ .

Hence  $G$  is a fixed point.

16. Let the polar of  $Q$  meet the conjugate  $CFD$  in  $R$ . Draw  $QQ'$  parallel to  $CP$ .

Then  $PE : PQ :: RF : PF$ ;

and  $PG : PQ :: CF : PF$ ;

hence  $EG : PQ :: CR : PF$ ;

or  $EG \cdot PF = PQ \cdot CR$ .

Now  $Q$  is on the polar of  $R$ , since  $R$  is on the polar of  $Q$ .

Hence  $PQ \cdot CR = CQ' \cdot CR = CD^2$ .

Hence  $EG$  is equal to the radius of curvature at  $P$ .

17. If  $O$  be the orthocentre of  $ABC$  and  $A'B'C'$  the reciprocal triangle,  $B'C'$ ,  $C'A'$ ,  $A'B'$  are perpendicular to  $OA$ ,  $OB$ ,  $OC$  respectively, and  $BC$ ,  $CA$ ,  $AB$  are perpendicular to  $OA'$ ,  $OB'$ ,  $OC'$  respectively.

Hence  $ABC$  and  $A'B'C'$  have their sides parallel and  $O$  is the orthocentre of each.

18. If  $pPSQq$  be the focal chord and the tangents at  $P$  and  $Q$  meet in  $T$ ,  $TS$  is perpendicular to  $PQ$ , hence the tangents at  $p$  and  $q$  meet in  $T$ .

19. This theorem is the reciprocal of the following: if two circles intersect they have two common tangents: if one circle lie entirely within the other, they have no common tangents. Reciprocate with respect to a point on one circle and within the other.

20. If  $I$ ,  $C$  be the centres of the inscribed and circumscribed circles, and  $CI$  meet them in  $r$ ,  $r'$  and  $B$ ,  $B'$  respectively, then if  $AA'$  be the major axis of the ellipse into which the circumscribed circle is reciprocated,

$$IA \cdot IB = Ir^2, \quad IA' \cdot IB' = Ir'^2,$$

$$CI^2 = CB^2 - 2CB \cdot Ir.$$

Hence  $IA : Ir :: Ir : CB - CI :: CB + CI : 2CB$ .

and  $IA' : Ir :: Ir : CB + CI :: CB - CI : 2CB$ .

Hence  $AA' : Ir :: 2CB : 2CB$ ;

or  $AA' = Ir$ .

21. The four circles which circumscribe the triangles of a complete quadrilateral meet in a point.

22. See Ex. 30 on the parabola, or by reciprocation.

23. See Ex. 46 on the ellipse, or by reciprocation.

24. Let  $CPP'$ ,  $COO'$  be perpendicular to the polars of  $P$  and  $O$ .

Draw  $OX$ ,  $OA$  perpendicular to  $CP$  and the polar of  $P$ ;

$PY$ ,  $PB$  perpendicular to  $CO$  and the polar of  $O$ .

Then  $CO' : CP' :: CP : CO :: CY : CX$ .

Hence  $CO' - CY : CP' - CX :: CY : CX :: CP : CO$

or  $PB : OA :: CP : CO$ .



25. See Ex. 18 on Chapter I.

26. (1) If a quadrilateral circumscribe a conic a pair of opposite sides subtend at the focus angles which are together equal to two right angles.

(2) If we reciprocate with respect to the focus  $S$  the new theorem is, if  $Q$  be taken on a circle and  $QL$  be drawn such that the angle  $SQZ$  is constant,  $QL$  envelopes a conic of which  $S$  is focus.

27. The envelope of chords of a circle which subtend a constant angle at a fixed point on the circle is a smaller concentric circle.

28. Two circles, such that a point can lie within both cannot have more than two common tangents.

But if the circles be such that all points lie without both, or within one and without the other they may have four common tangents.

29. If a straight line meet the sides of the triangle  $A'B'C'$  in  $L, M, N$  the circles circumscribing the triangles  $A'B'C', A'NM, B'NL, C'LM$  meet in a point.

30. If points  $P', Q'$  be taken on a circle of which  $C$  is the centre,  $P'C$  will meet the line drawn through  $Q'$  at right angles to  $P'Q'$  and  $Q'C$  will meet the line drawn through  $P'$  at right angles to  $P'Q'$  on the circle.

31. If  $S$  be the orthocentre of the triangle  $ABC$  and circles be described with centres  $A$  and  $B$  passing through  $C$ ,  $S$  will lie on the radical axis of the two circles.

If we reciprocate with respect to  $S$  we see that if with the orthocentre of a triangle as focus we describe two conics each touching a side of the triangle and having the other two sides as directrices, the conics will have a parallel pair of common tangents and therefore their minor axes equal.

32. If a system of circles have two points in common the locus of their centres is a fixed straight line, and the polar of a fixed point meets the radical axis in a fixed point.



33. If the tangent and normal at  $P$  meet  $QR$  in  $T$  and  $G$ , the range  $TRGQ$  is harmonic, since  $TP$ ,  $PG$  bisect the angle  $QPR$ .

Hence  $PG$  is the polar of  $T$ .

Hence the pole of  $QR$  lies on  $PG$  since the pole of  $PG$  lies on  $QR$ .

34. If the tangents at  $P$  and  $Q$  meet in  $T$  and  $TA$  meet  $PQ$  in  $L$ , the range  $DPLQ$  is harmonic ; hence the pencil  $TD$ ,  $TP$ ,  $TL$ ,  $TQ$  and the range  $DBAC$  are harmonic.

Therefore  $ABDC$  is a harmonic range.

35. If the pencil joining  $BPAQ$  to any point on the curve is harmonic, the pencil formed by joining them to any other point on the conic is harmonic.

For if  $BK$ ,  $PK$ ,  $AK$ ,  $QK$  meet the directrix in  $bpaq$ ,  $bpaq$  is a harmonic range, provided  $KEBPAQ$  be a harmonic pencil.

And the angles  $bSp$ ,  $pSa$ ,  $aSq$ ,  $qSb$  are half the angles  $BSP$ ,  $PSA$ ,  $ASQ$ ,  $QSB$  ;

Hence the pencil  $Sb$ ,  $Sp$ ,  $Sa$ ,  $Sq$  is the same wherever  $K$  be taken on the curve.

Now  $PQ$  goes through  $O$  the pole of  $AB$  : let  $PQ$  meet  $AB$  in  $R$ .

Then if  $T$  be the pole of  $PQ$ ,  $TARB$  is a harmonic range.

Therefore the pencil joining  $Q$  to  $BPAQ$  is harmonic ;

hence the pencil joining  $q$  to  $BPAQ$  is harmonic.

Hence  $Pq$  bisects  $AB$  since  $AB$ ,  $qQ$  are parallel.

36. Four circles can be described so as to touch the sides of a triangle, and the reciprocal of the radius of the inscribed circle is equal to the sum of the reciprocals of the radii of the other three.

If the triangle be equilateral the inscribed circle touches the three escribed circles.

37. If the tangents at  $P$  and  $Q$  meet the axes in  $T$  and  $V$ ; the angle

$$PSQ = SQV - SPT = SVQ - STP = VST.$$

If  $SW$  be perpendicular to  $P'Q'$ , the tangents at the vertices intersect in  $W$ .

Draw  $SYZ$  perpendicular to the tangents at  $P$  and  $Q$ .

Then  $WSP'$ ,  $WSQ'$  are supplementary to  $WYP'$ ,  $WZQ'$ .

Hence  $P'SQ'$ ,  $PSQ$  are supplementary.

If two circles intersect in  $P$ ,  $Q$  the angle between the tangent at  $P$ ,  $Q$  is equal to the angles which the centres subtend at  $S$  and supplementary to the angle which  $PQ$  subtends at the other point of intersection.

38 and 39. If from any point  $P$  in the radical axis tangents be drawn to the circles, and a circle be described, with centre  $P$  and radius equal to the tangent, this circle will intersect the line of centres in two points  $E$  and  $F$  which are the limiting points of the system.

Take  $A$  at centre of one of the circles, and  $M$  at the point where the radical axis intersects the line of centres.

Then,  $PU$  and  $PU'$  being the tangents from  $P$  to the circle,

$$PM^2 + ME^2 = PE^2 = PU^2 = PA^2 - AU^2;$$

$$\therefore ME^2 = AM^2 - AU^2 = MF^2;$$

$$\therefore AE \cdot AF = AM^2 - EM^2 = AU^2.$$

$\therefore$  the polar of  $F$  passes through  $E$ .

Reciprocating with regard to  $F$ , the pole of  $uU'$ , i.e. of the fixed line through  $E$ , is the centre, which is therefore fixed, and the conics are confocal.

Therefore, if we reciprocate with regard to either limiting point we obtain confocal conics.

40. If perpendiculars be drawn from  $A$ ,  $B$ ,  $C$  to  $BC$ ,  $CA$ ,  $AB$  these lines will meet in a point  $O$ , and the circles circumscribing  $ABC$ ,  $OBC$ ,  $OCA$ ,  $OAB$  are equal.

41. If the tangents at  $P$  and  $Q$ , points on a circle intersect at a constant angle, and lines be drawn through

$P$  and  $Q$  making constant angles with the tangents at  $P$  and  $Q$  respectively, this pair of straight lines will intersect on a concentric circle.

42. If two circles intersect in  $A$  and  $B$  and  $PQ$  be a common tangent and  $QB, PA$  meet the circles in  $C$  and  $D$ , then  $PC, QD$  are parallel.

43. If from any point on a circle circumscribing a triangle perpendiculars be drawn to the sides of the triangle, the feet of these perpendiculars lie on a straight line.

44. Since the orthocentre is on the hyperbola,  $DEF$  is a self-conjugate triangle and the pole of  $EF$  lies on  $BC$ .

Hence the pole of  $BC$  lies on  $EF$ .

45. If  $AB, CD$  meet in the fixed point  $E$ ,  $CA$  and  $BD$  in  $F$ , and  $BC$  and  $AD$  in  $G$ , then  $FG$  is the polar of  $E$ .

Hence the centre of the circle lies in a straight line through  $E$  perpendicular to  $FG$  the polar of  $E$  with respect to both curves.

46. The radius of an escribed circle of an equilateral triangle is  $\frac{2}{3}$  the radius of the circumscribed circle, and if  $SE$  be the tangent from the centre of the circumscribed circle to the escribed circle whose centre is  $D$ ;

$$SD = DE + \frac{1}{3}DE = \frac{4}{3}DE.$$

The proposition in the question is obtained by reciprocating with respect to the circumscribed circle.

47. If  $AD$  be drawn parallel to the axis to meet  $BC$ ,  $AD$  is bisected at  $D'$  where it meets the curve.

Hence the tangent at  $D'$  is parallel to  $BC$  and bisects  $AB$  and  $AC$ .

Since a straight line intersects a conic in two points and two tangents can be drawn from a point, the reciprocal polar of a conic with respect to another conic is a third conic.

Now by Ex. 44, if a rectangular hyperbola circumscribe a triangle  $DEF$  it will go through the ortho-centre  $O$  and  $ABC$  the triangle formed by joining the feet of the perpendiculars is a self-conjugate triangle, and  $O$  is the centre of the circle inscribed in  $ABC$ . If we reciprocate with respect to  $O$  the reciprocal conic is a parabola, since it has one tangent at an infinite distance and  $ABC$  is a self-conjugate triangle.

The tangents from  $O$  are at right angles, since the hyperbola was rectangular, hence  $O$  is on the directrix.

The locus of the poles of the lines at an infinite distance, that is, of the centres of the hyperbolas, was the circle circumscribing  $ABC$ .

Hence the envelope of the polars of  $O$  with respect to the parabolas is an ellipse inscribed in  $ABC$  having  $O$  for a focus. Since  $O$  is now the centre of the circle circumscribing  $ABC$ , the auxiliary circle of the ellipse is the nine point circle.

## MISCELLANEOUS PROBLEMS.

1. If  $S$  and  $H$  be the rifle and target, and  $P$  the hearer, the difference of the times in which sound travels from  $S$  and  $H$  to  $P$  is equal to the time of the bullet's transit from  $S$  to  $H$ .

Hence  $HP - SP$  is constant, and the locus is a hyperbola of which  $S$  and  $H$  are the foci.

2. Let  $tp$ ,  $tq$  be the tangents parallel to  $PQ$  and  $P'Q'$  and let  $qt$  meet in  $r$  the diameter through  $p$ ; then  $qt = tr$ , and

$$\begin{aligned} PR^2 : PQ^2 &:: tr^2 : tp^2 :: tq^2 : tp^2 \\ &:: SP' \cdot SQ' : SP \cdot SQ \\ &:: P'Q' : PQ; & \text{Art. 17.} \\ \therefore PR^2 &= PQ \cdot P'Q'. \end{aligned}$$

3.  $QN : CM :: BC : AC :: DM : CN$ ,  
hence  $QN + DM : NM :: BC : AC$ .

4. If  $CVED$  be conjugate to  $PQ$ ,  
 $PQ \cdot Qp = PV^2 - QV^2$ .

Hence  $PQ \cdot Qp : CD^2 - CE^2 :: CR^2 : CD^2$ ,  
 $CR$  being parallel to  $PQ$ .

5.  $QN : NX :: SA : AX :: SR : SX$ .

Hence  $Q$  lies on the tangent at the extremity of the latus rectum.

6.  $PN^2 : AN \cdot NA' :: BC^2 : AC^2$   
and  $QN \cdot PN = AN \cdot NA'$ .

Therefore  $QN^2 : AN \cdot NA' :: AC^2 : BC^2$ .



7. Let  $AP$ ,  $QB$  meet in  $R$ , and draw  $RV$  parallel to  $POQ$ .

Then  $RV : VA :: PO : AO$ ,

and  $RV : VB = QO : OB$ .

Hence  $RV^2 = VA \cdot VB$ ,

since  $AO \cdot OB = PO \cdot OQ$ .

Therefore  $QR$  lies on a concentric rectangular hyperbola.

8. The line joining  $T$  to the intersection of the normals at  $P$  and  $P'$  bisects  $PP'$  and therefore passes through the centre.

9. If the tangent at  $P$  meet the tangents at  $A$  and  $A'$  in  $T$  and  $T'$  and  $TS$ ,  $T'S'$  meet in  $Q$ ,

the angle  $SS'Q = T'S'A' = T'S'P$ ;

$$QSS' = AST = TSP;$$

and  $SPT = S'PT'$ .

Hence  $S$ ,  $P$ ,  $S'$  are the feet of the perpendiculars of the triangle  $TQT'$ .

Therefore  $QP$  is perpendicular to  $TP$ .

10. Make the angle  $PSF$  a right angle, then the tangent at  $P$  meets the directrix in  $F$ : if a circle be described with centre  $P$  and radius  $PK$  such that the ratio  $SP : PK$  is equal to the eccentricity the directrix is a tangent from  $F$  to this circle.

Two tangents can in general be drawn.

If the angle  $SPF$  be such that  $SP : PF :: SA : AX$ , only one conic can be constructed; there are two positions of  $PF$  equally inclined to  $SP$  corresponding to this case.

If the eccentricity be unity, one conic is a line parabola through  $S$ .

11. If  $PK$  be drawn perpendicular to the directrix of the parabola  $SP = PK$ ,

hence  $HM = HP + PK = AA'$ .



Therefore the directrix touches a circle of which  $H$  is centre.

12. Draw  $RN$  perpendicular to the minor axis.

Then

$$CN.AC = SR.AC = BC^2 = AC^2 - SC^2 = AC^2 - RN^2;$$

Hence  $RN^2 = AC\{AC - CN\},$

or  $R$  lies on a parabola.

13. If  $ST, PQ$  meet in  $H$ ,  $HTRS$  is a harmonic range.

But  $OH, OT, OV, OS$  is a harmonic pencil.

Hence  $OV$  passes through  $R$ .

14. Let  $ACA'$  be the diameter bisecting the parallel chords  $QN$ , etc. in  $N$ , etc.

Then  $PN^2$  varies as  $QN^2$ , that is as  $AN.NA'$ .

Hence the locus of  $P$  is an ellipse. The locus will be a circle if  $PN = QN$ , that is if the vertical angle is a right angle.

15. If  $PP', QQ'$  be the double ordinates of the given points,  $P, P', Q, Q'$  are fixed points, and since the ellipses are similar, the corresponding points of the auxiliary circle, at which the major axis subtends a right angle, are likewise fixed points.

16. The ordinates of the point and of the end of one of the radii are in the ratio of the radii.

17. If  $P$  be the centre of the circle and  $PK$  perpendicular to the fixed straight line, the ratio  $SP : PK$  is constant.

18. Let  $pqr$  be a triangle touching the parabola in  $P, Q, R$ .

Parabolic area  $PQR = \frac{2}{3}$  triangle  $PqR$ .

$$\therefore \text{Triangle } PQR = \frac{2}{3}(PqR - PrQ - QpR),$$

$$3PQR = 2(pqr + PQR);$$

$$\therefore PQR = 2pqr.$$

19. If a rectangular hyperbola circumscribe a triangle the orthocentre is on the curve, if we reciprocate with respect to the orthocentre we have the case of a parabola inscribed in a triangle the tangents from the orthocentre being at right angles. See Ex. 47, Ch. 10.

20. Produce  $HA$  to  $K$  making  $AK$  equal to  $AH$ , then  $PK$  and  $AL$  are parallel.

Hence  $SQ : SP :: SA : SK :: SA : SA + AH$ .

Therefore the locus of  $Q$  is a similar ellipse of which  $S$  is focus.

21. If  $KLMN$  be the quadrilateral and  $k, l, m, n$  the points of contact,  $KM$  will bisect  $nk, lm$ : and  $LN$  will bisect  $kl, mn$ .

Hence  $klmn$ , and therefore  $KLMN$ , is a parallelogram.

22. The locus of the second focus is a circle of which the radius  $= AA - SP$ .

The locus of the centre which bisects  $SH$  is similar, that is, a circle.

23. Since  $LC, LL'$  are tangents, the angles  $HLC$  and  $SLL'$  are equal.

Again  $CL \cdot CL' = CH^2$ ,

hence the angle  $CHL = CL'H = SL'L$ .

Therefore  $CL : HL :: SL : LL'$ ,

the triangles  $CLH, SLL'$  being similar.

24. If the theorem be true in the case of a circle, it will follow by orthogonal projection for any ellipse.

If  $PQ$  be the chord of contact of tangents drawn to a circle from a point on a concentric circle, the angles  $PAQ, PA'Q$  will be constant,  $A, A'$  being extremities of a fixed diameter.

Let  $AP, A'Q$  meet in  $R$ , and  $A'P$  and  $AQ$  in  $R'$ .

The angle  $ARA' = APA' - PA'Q$ ,

and the angle  $ARA' = APA' + PAQ$ .

Hence the loci of  $R$  and  $R'$  are circles passing through  $A$  and  $A'$ .

25. Circumscribe circles to two of the triangles formed by the intersections of the tangents, these circles intersect in the focus  $S$ : the pedal line of  $S$  is the tangent at the vertex.

A parabola can be drawn to touch five straight lines, if the circles circumscribing the triangles formed as above all meet in the same point  $S$ .

26.  $PF$  is the same for both curves, and therefore  $CD$  is also the same.

27. Prove that  $SY \cdot S'Y'$  is constant.

28. By reciprocation.

29.  $P, Q, R, R'$  lie on a circle of which  $PQ$  is diameter and  $PQ, LL'$  are equally inclined to the axis. If  $p, p'$  are the vertices of the diameters bisecting  $PQ, RR'$  in  $V$  and  $V'$ ,  $pp'$  is a double ordinate.

Let  $VV'$  which is parallel to the normal at  $p'$  meet the axis in  $O$ .

Let  $VM, V'M'$  be the ordinates of  $V$  and  $V'$ .

Then  $LL' = L'M' + M'O + MO - LM = 2OM = 2ng = 4AS$ .

30. The bisectors are tangent and normal to a confocal conic.

Hence  $CG \cdot CT = CS^2$ .

31. Reciprocate the following theorem: if  $S, A, B, C$  be points on a circle and with centres  $A, B, C$  and radii  $AS, BS, CS$  circles are described, they will intersect two by two in points which lie in a straight line.

32.  $OE : EG = SP : SG = Sp : Sg = OE : Eg$ .

33. If an ellipse be reciprocated with respect to its centre, the reciprocal is a similar ellipse having its major axis in the minor axis of the original ellipse.

If we reciprocate an ellipse circumscribing a triangle and having its centre at the orthocentre with respect to that orthocentre, the reciprocal is a similar ellipse, inscribed in a triangle having its sides parallel to those of the original triangle, the homologous axes being at right angles, and having its centre at the orthocentre.

This reciprocal ellipse is similar and similarly situated to the ellipse inscribed in the original triangle having its centre at the orthocentre.

34.  $Q$  lies on the common circle of curvature,  
hence  $PQ = 4PT$ .

35.  $AB, BC$  are equally inclined to the axis, hence since the angles at  $A$  and  $C$  are equal,  $AD, DC$  are equally inclined to the axis.

Hence the tangent at  $D$  and  $AC$  are equally inclined to the axis.

Therefore the tangents at  $B$  and  $D$  are parallel.

36. The volume cut off varies as the area  $VAA'$  and  $BB'$ ; and the area  $VAA'$  varies as  $AV \cdot VA'$  or  $AD \cdot A'D'$ , that is  $BC^2$ .

37.  $SQ : Pg :: St : tg :: SY : SP :: BC : CD$   
 $:: PF : AC$ .

Hence  $SQ \cdot AC = PF \cdot Pg = AC^2$ ,

or  $AC = SQ$ .

Let  $QL$  be the ordinate of  $Q$  and let  $MQ$  meet the major axis in  $V$ .

Then  $CV : PN :: CV : CM :: CL : CM - QL$   
 $:: CL : PN - QL$ ,

and  $SP - AC : CN :: SC : AC$ ,

or  $AC \cdot SP = AC^2 + CN \cdot SC = BC^2 + CS \cdot SN$ .

Again  $CN - CL : SP - SQ :: SN : SP$ ,

and  $SP - SQ : CN :: SC : AC$ ;

hence  $CN - CL : CN :: SN \cdot SC : SP \cdot AC$ ;

therefore  $CL : CN :: BC^2 : SP \cdot AC$ ,

or  $CL : SP - SQ :: BC^2 : SP \cdot SC$ .

Now  $SP - SQ : PN - QL :: SP : PN$ ,

hence  $CL : PN - QL :: BC^2 : PN \cdot SC$ .

Hence  $CV \cdot SC = BC^2$ ,

or  $V$  is a fixed point.

38. If  $SL$ ,  $SM$ ,  $SN$  be drawn perpendicular to the given tangents, the circle circumscribing  $LMN$  is the auxiliary circle of the conic.

39. If  $S$  be the centre of the circumscribed circle,  $H$  the orthocentre, the centre of the nine-point circle bisects  $SH$ : and if  $PQR$  be the triangle, the angles  $SPQ$ ,  $HPR$  are equal: hence  $S$  and  $H$  are the foci.

40. If  $Pg$ ,  $P'g'$  be the normals at  $P$  and  $P'$  and  $gL$ ,  $g'L'$  be drawn perpendicular to  $PP'$ ,

then  $PL = P'L'$ , by Ex. 27, Chapter I.;

hence  $gG = Gg'$ .

Therefore

$$2SG : SP + SP' :: Sg + Sg' : SP + SP' :: SA : AX.$$

41. Reciprocate the following with respect to  $S$ :  $ASB$  is a diameter of circle meeting a concentric circle in  $S$ , the opposite sides of the quadrilateral formed by tangents through  $A$  and  $B$  to the inner circle are parallel, and the tangents to the outer circle at the points where it meets the tangent at  $S$  are respectively parallel to them.

42. If  $P$ ,  $Q$  be two points on a rod and  $PS$ ,  $QS$  are at right angles to the directions of motion of  $P$  and  $Q$ , then if  $R$  be any point on the rod the direction of motion of  $R$  is at right angles to  $SR$ .

Hence the directions of motion of all points on the rod envelope a parabola of which  $S$  is focus and the rod tangent at the vertex.



43.  $SP$  and  $HQ$  are parallel to  $CT$ .

Hence  $PSp = \text{complement of } SPp = \text{complement of } CTP$ .

$QHq = \text{complement of } HQq = \text{complement of } CTQ$ .

Hence  $PSp$  and  $QHq$  are together equal to the supplement of  $PTQ$ .

44. Let  $ST, S'T$  meet  $CP$  in  $p$  and  $p'$ , and  $CD$  in  $d$  and  $d'$ .

Since the angle  $PTS = d'TD$ ,

and  $TPp = TDd'$ ,

the angles  $SpC, Cd'S'$  are equal, and  $d, d', p$  and  $p'$  lie on a circle.

45. If the asymptote and directrix meet in  $D$ ,  $SDC$  is a right angle and if  $DP$  be the tangent  $PSD$  is a right angle.

Therefore  $SP$  is parallel to the asymptote.

46. If  $PTQ, ptq$  be two consecutive positions and  $TV, ty$  be drawn perpendicular to  $pt, TQ$  respectively,

$$tV = Pp + pt - PT = PT + TQ + Qq - tq - PT \\ = TQ + Qq - tq = Ty.$$

Hence the tangent at  $T$  is equally inclined to  $PT$  and  $TQ$ .

Hence  $T$  lies on a confocal ellipse.

47.  $CP$  and  $CQ$  are at right angles ; hence  $C, P, D, Q$  lie on a circle.

48. If  $PV$  be the chord through the centre, and  $pp'$  the parallel focal chord,

$$PV \cdot CP = 2CD^2,$$

and

$$pp' \cdot CA = 2CD^2.$$

Hence

$$PV : pp' :: CA : CP.$$

49.  $SP$  and  $QH$  are both parallel to  $CT$  ;  
hence the angle

$$pCq = pCT + TCq = PHQ + QSP.$$

50. The angle  $SPY = \text{half the supplement of } SPS' = \text{half } PSP$ .



51.  $QSP$ ,  $Q'SP'$  are right angles and the perpendiculars from  $Q$ ,  $Q'$  on  $SP$ ,  $SP'$  are equal to the perpendiculars on  $PP'$ .

Hence  $QP$ ,  $Q'P'$  are tangents at  $Q$  and  $Q'$  to the parabola of which  $S$  is focus and  $PP'$  directrix.

And the diameter parallel to  $PP'$  is tangent at the vertex, since it bisects  $SH$ .  $BB'$  is a tangent since  $SCB$  is a right angle.

52.  $SE$  will evidently envelope a conic of which  $S$  is focus and the given circle auxiliary circle.

53. Join  $SP$ , the bisector of  $POS$  bisects  $SP$  in  $V$ ; since the locus of  $P$  is a circle, the locus of  $V$  is a circle.

Hence  $VO$  envelopes a conic of which  $S$  is a focus.

54. The angles  $RSP$  and  $QHV$  are the complements of  $PST$  and  $QHT$ .

Hence  $RSP + QHV = \text{supplement of half } PSQ + PHQ = \text{half the angle between the tangents at } P \text{ and } Q$ , by Ex. 23 on the ellipse.

55.  $TS$ ,  $ZS$  are the interior and exterior bisectors of the angle  $QSR$ ,

and if  $TQ$  meet  $PZ$  in  $F$ ,  $FS$  is the exterior bisector of the angle  $QST$ .

Hence  $Q$ ,  $T$ ,  $R$  lie on a conic of which  $S$  is focus and  $PZ$  directrix and  $ZT$  is the tangent at  $T$  since  $ZST$  is a right angle.

56. If  $OE$  be the radius of the sphere,

$OE \cdot AC = \text{area } OAA'$

and varies as  $OA \cdot OA'$  or  $AD \cdot A'D'$  that is as  $BC^2$ .

Hence the latera recta of all the sections are the same.

57. Let  $PQ$ ,  $QP'$  meet the ellipse in  $U$  and  $V$ ,

then since  $TP^2 = TQ \cdot TQ'$

and  $TQ^2 = TP \cdot TP'$ ,

$$TP : TP' :: TQ' : TQ,$$

or  $PQ$ ,  $P'Q$  are parallel.

Hence if  $CF$  be parallel to  $P'Q$  and  $PQ$ ,

$$P'V.P'Q : P'P^2 :: CF^2 : CD^2,$$

and  $QU.QP : QQ^2 :: CF^2 : CE^2.$

Now  $P'P^2 : QQ^2 :: TP^2 : TQ^2 :: TQ : TQ',$

and  $CE^2 : CD^2 :: TQ^2 : TP^2 :: TQ : TQ'.$

Hence  $P'V.P'Q : QU.QP :: TQ^2 : TQ'^2 :: P'Q^2 : PQ'^2.$

Therefore  $PV : P'Q :: QU : QP.$

58. If the tangents at  $P$  and  $Q$  meet in  $T$ ,  $CT$  which bisects  $PQ$  is parallel to  $P'Q$ ,  $P'$  being the other extremity of the diameter  $PCP'$ .

Hence the angle  $PCT = PP'Q = TPQ.$

59.  $S, P, S', g$  lie on a circle,

hence  $Sg : Pg :: S'G : S'P :: SA : AX.$

60.  $BC$  is parallel to the polar of  $A$ , hence  $AD$  is parallel to the axis.

Also the angles  $SAC, DAB$  are equal: and the angles  $ABD, ASC$  are likewise equal.

Therefore  $AC : AS :: AD : AB.$

61.  $RK : QN :: KC : NC$

and  $PN : RK :: NG : KG.$

Hence  $BC : AC :: KC.NG : NC.KG,$

or  $KC : KG :: AC : BC.$

Therefore  $CR : CQ :: CK : CN :: AC + BC : AC,$

or  $CR = AC + BC.$

Hence if  $NP$  meet  $RL$  in  $N'$ ,  $PN = QN.$

Hence  $KL$  passes through  $P.$

Also  $PL = QC = AC,$

and  $KP = QR = BC.$

62.  $CT.CN = CA^2$

and  $CT'.PN = BC^2;$

hence  $CT.CT' : CA.CB :: CA.CB : CN.PN$ .

Hence the triangle  $CTT'$  varies inversely as the triangle  $PCN$ .

63. By Ex. 6. Chap. VII.

$$2SP = 3AC,$$

and if  $CE$  be drawn parallel to the tangent at  $P$ ,

$$PE = AC.$$

64. Let  $CT$  which bisects  $PP'$  in  $V$ , meet the ellipse in  $Q$ , and let  $CE$  be conjugate to  $CQ$ .

Then  $PV^2 : CV.VT :: CE^2 : CQ^2 :: PV^2 : CQ^2 - CV^2$ .

Hence  $CQ^2 = CV.VT + CV^2 = CV.CT$ ,

or  $TP, TP'$  are tangents.

65. See Ex. 82 on the hyperbola.

66. If we reciprocate with respect to a focus the theorem that tangents to an ellipse at right angles intersect on a fixed circle, we find that if the sides of a quadrilateral  $ABCD$  subtend each a right angle at a fixed point  $S$  the sides envelope an ellipse of which  $S$  is a focus. If  $O$  be the centre,

the angle  $OAB =$  complement of half  $AOB =$  complement of  $SCB = CBS = SAD$ .

Hence  $O$  is the other focus.

67. If  $A', B', C', D'$  be the points of contact and  $E', F', G'$  the points of intersection of  $A'C', B'D'; A'D', B'C'; A'B', C'D'$ , then  $E'F'G'$  is a self-conjugate triangle.

If  $BA'A, C'DF$  meet in  $F$  the pole of  $A'C'$ ,  $F$  will lie on  $F'G'$  the polar of  $E'$ , since  $E'$  lies on  $A'C'$  the polar of  $F$ .

Similarly  $AD'D, BB'C$  meet in  $G$  on  $F'G'$  and  $AC, BD$  in  $E'$ .

Let  $A'D', CC'D$  meet in  $\alpha$ , and  $B'C', AD'D$  in  $\beta$ .

Then if  $O$  be intersection of  $AC', CD'$ ,  $\alpha\beta$  is the polar of  $O$ .

Now since  $GB'$ ,  $GD'$  are tangents and the tangent at  $C'$  meets them, the range  $CC'DF$  is harmonic and therefore the pencil  $GB'$ ,  $GC'$ ,  $G\beta$ ,  $GF'$ .

Hence  $B'C'\beta F'$  is a harmonic range.

So  $A'D'aH$  is a harmonic range.

Hence  $a\beta$  passes through  $G'$ .

Since  $G'$  lies on the polar  $O$ ,  $O$  lies on  $BD$ , the polar of  $G$ . Similarly  $AB'$ ,  $CA'$  intersect on  $BD$ .

68. By Art. 138 the four points in which two rectangular hyperbolas intersect are such that any one of them is the orthocentre of the triangle formed by the other three: hence any conic through the four points is a rectangular hyperbola.

69. If  $KVt$  be drawn parallel to the axis to meet,  $PQ$ ,  $ST$  in  $V$  and  $t$ , and if  $KL$  be perpendicular to  $PQ$ ,  $PL = SQ$  and  $PV = VQ$ , hence  $tV = VK$ . Therefore  $ST$   $SP$ ,  $SK$  and the axis form a harmonic pencil.

70. Let  $DE$  meet  $PP'$  in  $K$ ; and let  $PD$ ,  $P'E$  meet in  $H$ .

Then  $GH$  is the polar of  $K$ , but  $K$  lies on  $DE$  the polar of  $F$ .

Hence  $F$  lies on  $GH$ , and  $FG$  is parallel to the chords bisected by  $PP'$ .

71. Let  $RR'$  the common tangent be bisected by  $PQ$  the common chord in  $O$ .

Then  $RO^2 = OR'^2 = OQ \cdot OP$ ;

hence  $RR'$ ,  $PQ$  are equally inclined to the axis.

Hence  $PR$  is a diameter, and the diameter of curvature  $= 2PF = 2CD$ .

Therefore  $CD^2 = CD \cdot PF = AC \cdot BC$ .

72. Reciprocate with respect to  $S$  the following theorem:  $S$  is taken on the outer of two concentric circles;  $SY$ ,  $SZ$  are drawn perpendicular to a pair of parallel tangents to the two circles;  $YZ$  is constant.

73. Let the tangents at  $P$  and  $Q$  meet in  $T$ .

Then the angle  $P'RQ' = PTQ =$  supplement of  $PCQ$ , by Ex. 58.

Hence  $P', C, Q'$  and  $R$  lie on a circle.

74.  $TN$  is half the difference of  $QM$  and  $Q'M'$  and  $RR'$  is half their sum.

Hence  $RP = \frac{1}{2} Q'M' = KR$ .

Now  $PN : Q'M' :: TK : 2KR :: PM : QM :: QM : 4SP$ .

Hence  $PN : QM :: Q'M' : 4SP :: PM' : Q'M'$ .

Therefore  $PN^2 : PM'^2 :: QM^2 : Q'M'^2 :: PM : PM'$ ,  
or  $PN^2 = PM \cdot PM'$ .

75. Let  $RR'V$  be the diameter bisecting  $PQ$ .

Then if  $PQ$  meet a common tangent  $pp'$  in  $O$ ,

$Op^2 : OP \cdot OQ :: Sp : SR :: S'p' : S'R' :: Op'^2 : OP \cdot OQ$ ,  
or  $Op = Op'$ .

76. Let  $O, C$  be the centres of the hyperbola and ellipse to  $tbCb't'$  the tangent at  $C$ , then since  $PN \cdot Ct = Cb^2$  and  $CN \cdot CO = Ca^2$ , the area of the ellipse will be a maximum, when  $CN \cdot PN$  is maximum, that is, when  $CN \cdot NO$  is a maximum, or  $ON = NC$ .

Hence  $PP'$  is a tangent to a similar hyperbola.

77.  $RP \cdot RP' = RN^2 - PN^2 = RN^2 - 4AS \cdot AN$ .

Now  $RN^2 : 4AS \cdot AA' :: AN^2 : AA'^2$ ,

or  $RN^2 : 4AS \cdot AN :: AN : AA'$ .

Hence  $RN^2 - PN^2 : 4AS \cdot AN :: A'N : AA'$ .

Therefore  $RP \cdot RP' : AN \cdot A'N :: 4AS : AA'$ .

78. Let the tangent at  $P$  meet the confocal conic in  $Q$ .

Draw  $CEF$  parallel to  $PQ$  meeting the normal at  $P$  in  $F$ .

Then  $OP \cdot PF = CD^2 = SP \cdot PS' = Cd^2$ ,

$Cd$  being conjugate to  $CP$ .

Hence  $O$  is the pole of  $PQ$  with respect to the confocal.



79. Let the tangents meet in  $U$ ,  $SU$  meeting the curve in  $Q$ , and let the tangent at  $Q$  meet  $RR'$  in  $T$ . Then,  $V$  being point of contact of  $RR'$ ,

$$TSR = TSQ + USP - RSP = TSV + USP - RSV \\ = USP - RST,$$

$$\therefore 2TSR = USP = RSR';$$

$\therefore$  locus of  $T$  is tangent at  $Q$ .

Or by reciprocation of the theorem,

If  $ABC$  be a triangle inscribed in a circle, and  $DE$  the diameter perpendicular to  $AC$ ,  $DB$  and  $EB$  bisect the angle  $B$  and its supplement.

80. Reciprocate with respect to any point  $S$  the theorem that if two points on a circle be given, the pole of  $PQ$  with respect to that circle lies on the line bisecting  $PQ$  at right angles.

$$81. \quad PQ \cdot PR = PE \cdot PF = AC \cdot BC$$

$$\text{and} \quad QR = CE = AC - BC.$$

$$\text{Hence} \quad PQ = BC \text{ and } PR = AC.$$

$$\text{Also} \quad ER \text{ is parallel to } CQG.$$

$$\text{Hence} \quad PG : CD :: PG : PE :: BC : AC$$

$$\text{and} \quad PG \cdot PF = BC^2.$$

Hence  $CQ, CR$  are the axes.

82. This is a particular case of Art. 195, since the second point where  $AE$  meets the curve is at an infinite distance, hence  $AE = EK$ .

83. The circle of curvature is greatest at the extremity of the minor axis.

Hence  $BO$  the direction of the minor axis is given.

And  $BC \cdot BO = AC^2 = SB^2$ ,  $O$  being the centre of curvature.

Hence  $S$  lies on the circle of which  $BO$  is diameter.

$$84. \quad Ca : Cb :: Ba \cdot Ac : bA \cdot cB$$

$$\text{and} \quad Ca : Cb' :: Ba \cdot Ac' : b'A \cdot c'B,$$

by Todhunter's Euclid, Art. 59.



And  $Ac.Ac' : Ab.Ab'$  in ratio of squares on parallel diameters.

Hence  $BC$  is the tangent at  $a$ .

85. This depends on the fact that any chord is bisected by the diameter through the intersection of the tangents at the ends of the chord.

86. Let  $P, Q$  be the points of contact of parallel tangents to the conic and circle.

Then by Art. 132, the angles  $PCA, QCA$  are equal.

87. Let  $CL, CL'$  be the fixed straight lines,  $S$  the fixed point.

Then the angle  $LSL' = CSL' + CL'S$ ,

hence the angles  $CL'S, CSL$  are equal.

Therefore  $CL : CS :: CS : CL'$ ,

or  $LL'$  touches the hyperbola of which  $CL, CL'$  are asymptotes and  $S$  focus.

88. Since the semivertical angles are complementary they touch one another along their common generating line.

Now  $EA : AX :: CE : EO :: OE' : C'E' :: EA : AS'$ .

Hence  $S'$  coincides with  $X$ , and similarly  $S$  with  $X'$ .

89. Draw  $CRV$  perpendicular to the tangent at  $P$ ,  $CE$  bisecting  $PP'$  and  $PN$  parallel to  $CE$ .

Then  $QO.OQ' : PO.OP' :: CD^2 : CR^2 :: PO.PF : CN.CV :: PO : PE :: 2PO : PP'$ .

90. A circle can be described with centre  $T$  to touch  $SP, SQ, HP, HQ$ .

Hence  $SN - NH = SM - MH$ ,

and  $TM, TN$  bisect the angles at  $M, N$ .

Hence  $TM, TN$  touch a confocal conic passing through  $M$  and  $N$ .

91. If  $S$  and  $H$  are the given points, the locus of  $P$  is the conic in which the given plane through  $S$  intersects the

surface generated by the revolution about  $SH$  of a conic of which  $S$  and  $H$  are foci.

If  $ST$  be drawn perpendicular to the plane to meet the directrix plane corresponding to  $S$  in  $T$ , the cone formed by joining  $H$  to all points of the locus of  $P$  is a right circular cone of which  $TH$  is axis.

$$92. \quad PG \cdot Pq = BC^2 = PQ^2;$$

$\therefore PGQ$  and  $PgR$  are similar triangles.

$$93. \quad \text{If } OL \text{ be the ordinate of } O,$$

$$LG : GN :: OL : P'N :: CL : CN,$$

$$\text{or} \quad LC : LG :: CN : NG :: AC : BC^2.$$

$$\text{Hence} \quad CL : CG :: AC^2 : AC^2 + BC^2.$$

$$\text{Therefore} \quad CO : CP' :: CL : CN :: AC^2 - BC^2 : AC^2 + BC^2.$$

94. The polygons  $V'SPV$  and  $Z'HPZ$  are similar and the perpendicular from  $C$  on  $VZ$  bisects  $VZ$ ;

hence if  $VE$  be taken on  $VV'$  such that  $VE = ZZ'$ ,

$$\text{then} \quad CE = CV' = CZ'.$$

$$\text{Hence} \quad VV' \cdot ZZ' = VV' \cdot VE = VC^2 - V'C^2 = CA^2 - CA'^2.$$

$$95. \quad \text{The centre is the middle point of } CP.$$

$$96. \quad TSQ \text{ and } TS'Q \text{ are right angles ;}$$

$\therefore$  the middle point of  $TQ$  is the centre of the circle  $TSQS'$  and is equidistant from  $S$  and  $S'$ .

$$97. \quad TQ : TP :: SQ : ST,$$

$$\text{and} \quad T'P : T'Q' :: ST' : SQ';$$

$$\therefore TQ \cdot T'P : TP \cdot T'Q' :: SQ \cdot ST' : SQ' \cdot ST \\ :: SQ \cdot PT' : SQ' \cdot PT.$$

98. Draw  $NE$  perpendicular to  $NM$ , and prove that  $E$  is a fixed point in the axis.

99.  $P$  and  $Q$  are equidistant from the plane of the circular section of the cone, which contains the centre of the section.

$$100. \quad \text{Produce } OC \text{ to } E \text{ so that } CE = OC;$$

then  $PE$  is parallel to  $CZ$  and  $EPY = CZO = OPY$ ; that is, the tangent to the curve bisects the angle  $OPE$ .

101. If  $PEQ$  be one of the tangents, and  $ERV$  the chord,

$$EP^2 : ER \cdot EV :: EQ^2 : ER \cdot EV,$$

for each ratio is that of the parallel focal chords.

102. If  $FE, FG$  be the tangents,  $F(TEQG)$  is harmonic, and  $EFG$  is a right angle.

103. Reciprocate with regard to  $C$  the theorem, that, if a circle centre  $C$  intersect another circle at right angles at the point  $E$ , and  $CPQ$  be any chord,  $CE^2 = CP \cdot CQ$ .

104. Reciprocate the conics into two intersecting circles.

105. Reciprocates into the theorem of the existence of the director circle.

106. If  $PEQ$  be the chord required, and  $P'EQ'$  a consecutive chord, the areas  $PEP', QEQ'$  are ultimately equal, and  $E$ , which is the centre of curvature, is the middle point of  $PQ$ .

$PQ$  is therefore the diameter of curvature and is inclined to the axis at the same angle as the tangent, i.e. half a right angle.

107. The pole  $F$  of the straight line is fixed, and  $P$ , the point of contact of a tangent, is the foot of the perpendicular from  $F$  on the normal.

108. The angle  $MNC = LCN = LCN$ , if  $l$  be the point where the tangent meets the other asymptote.

$\therefore MN$  is parallel to  $Cl$ , and passes through  $P$  the middle point of  $Ll$ .

109. The diameter of curvature being the same for both, it follows that  $SP : S'P$  is a constant ratio.

110.

$$CK'^2 = CP^2 + PK'^2 + 2PK' \cdot PF = AC^2 + BC^2 + 2AC \cdot BC;$$

$$\therefore CK' = AC + BC.$$

111. If  $P, V, R, Q$  be the points of contact of  $AB, BC, CD, DA$ ,

$$2ASB = PSV + PSQ,$$

and two right angles

$$= PSV + ASP + VSC = PSV + ASQ + CSR;$$

$$\therefore PSV = QSR = 2QSD,$$

and

$$2ASB = 2QSD + PSQ = 2ASD.$$

$$\therefore ASB = ASD = \text{a right angle}.$$

Also

$$ASP + DSR = ASD,$$

$\therefore PSQ$  is a straight line and  $PA, RD$  intersect on the directrix.

112. If the tangent at  $P$  meet the director circle in  $R$  and  $T$ , perpendiculars to the tangent through  $R$  and  $T$  are tangents to the ellipse.

Draw  $PE$  parallel to  $CR$ , meeting  $CT$  in  $V$  and take  $CQ^2 = CV \cdot CT$ ; similarly find the point  $D$  on  $CR$ ;

then  $CQ$  and  $CD$  are conjugate diameters, and the construction is completed in Art. 216.

113.  $Q'R'$  meets the axis in  $T$ , the pole of  $NP$ ;

$\therefore$  the tangents at  $Q', R'$ , meet at a point  $E$  on  $NP$ .

Let  $CE$  meet  $Q'R'$  in  $Y$ ;

$$\begin{aligned} \text{then } EN \cdot PN &= EN^2 - EN \cdot EP = EC^2 - CN^2 - CY \cdot CE, \\ &= EC \cdot CY - CN^2 = CA^2 - CN^2 = P'N^2. \end{aligned}$$

$$\therefore EN : P'N = P'N : PN = AC : BC = QM : QM,$$

and the tangent at  $Q$  passes through  $P'$ .

114. If  $S'$  be the other focus of the fixed ellipse, and  $H$  of the moving ellipse,

$$S'P = HP \text{ and } S'Q = HQ.$$

Join  $S'H$  meeting the chord in  $Z$ , and let fall  $SY$  the perpendicular on the chord;

then  $SY \cdot S'Z = SY \cdot HZ = BC^2$ , and in the chord touches a confocal conic.

115. Let  $O$  be the centre of the circle,  $PQ$  a chord of intersection not perpendicular to the axis, meeting an asymptote in  $L$ , and the axis in  $K$ .

$$\begin{aligned}\text{angle } EOC &= 90^\circ - LKC = 90^\circ - LCA + CLK \\ &= 45^\circ + CLK = LCA + LCE = ECO,\end{aligned}$$

$\therefore ECO$  is isosceles, and  $E$  lies on a fixed line perpendicular to the axis.

116. Project the ellipse into a circle, and prove that the angle  $DQP = QPR$ , observing that the tangent and common chord are equally inclined to the axis.

117. Reiprocates into the following :

If  $S$  be a fixed point, and  $SK$  a tangent to a circle centre  $C$ , and if  $TE$  be any other tangent from a point  $T$ , and the angle  $CTE = CSK$ , the locus of  $T$  is a circle passing through  $S$ .

$$\begin{aligned}118. \quad SY, HZ \text{ being perpendiculars on the tangent,} \\ Pq : HZ :: PT : TZ :: TR : TC :: PR : PE; \\ \therefore Pq : PR :: HZ : AC :: HP \cdot BC : AC \cdot CD \\ \qquad \qquad \qquad :: HP \cdot PG : CD^2 :: PG : S'P;\end{aligned}$$

$\therefore Rq$  is parallel to  $SG$ .

119. If  $PT, QT$  and  $pt, qt$  be two near positions and  $tM, Tm$  be drawn perpendicular to  $PT, Qt$ , then  $tM : Tm$  in the ratio compounded of  $PT : QT$  or  $CD : CE$  and  $Pp \cdot QO' : Qq \cdot PO$  or  $PF : QF' : QO', PO$  being the radii of curvature.

Hence  $tM = Tm$ , or the normal at  $T$  to the locus of  $T$  bisects  $PTQ$ .

Therefore  $T$  lies on a confocal ellipse.

120. Referring to the figure of Art. 148, and drawing the lines, a circle can be drawn through  $ADOG$ ;

$$\therefore \text{angle } DOA = DGA = 90^\circ - GVA = AEC,$$

and the triangles  $AOD, ACE$  are similar.

$$\therefore AO : AD :: CE : AC,$$

or

$$AO \cdot AC = AD \cdot A'D' = BC^2.$$



121. Referring to the preceding theorem, describe a sphere centre  $G$  and radius  $GA$ ; the tangent from any point of the ellipse to this sphere will be equal to the tangent from  $P$  to the circle of curvature.

Describing a similar sphere with centre  $G'$ , the sum of the tangents  $= FA' = SS'$ .

122. In the second figure of Art. 144, take  $T$  any point in the tangent at  $P$  and let  $C$  be the centre of the upper sphere.

Then  $CRP$  and  $CSP$  are right angles,  $PR = PS$ , and  $TR = TS$  these lines being tangents.

$\therefore T$  lies in the plane through  $CP$  perpendicular to the plane  $CRPS$ ;

$\therefore$  angle  $SPT = RPT$ , and  $RTP = STP$ ;

similarly  $R'TP = S'TP$ .

Hence  $RTR' = STP + S'TP = STP + STQ = PTQ$ ,  
 $TQ$  being the other tangent.

123. Produce  $ER$  to  $E'$  making  $RE'$  equal to  $RE$ .  
 Then the polar of  $E'$  passes through  $E$ .

Now  $C$  is the pole of  $PQ$  which passes through  $E$ .

Hence  $CE'$  is the polar of  $E$  and is therefore parallel to  $ARB$ .

Hence  $CE$  is bisected by  $AB$ .

Again  $CE$  bisects in  $E$  the polar of  $E'$  which is parallel to  $AB$ .

Therefore  $CE$  bisects  $AB$ .

Therefore  $ACBE$  is a parallelogram.

124. Let  $O$  be the centre of the conic which touches the sides  $AB, BC, CD, DA$  in  $E, F, G, H$ .

Then since  $OA, OB, OC, OD$  bisect  $HE, EF, FG, GH$  the sum of the areas of the triangles  $AOB, COD$  is half the area of the quadrilateral.



Let  $AB, CD$  meet in  $K$  and let  $O, O'$  be two positions of  $O$ .

Draw  $OM, O'M'$  perpendicular to  $AB$  and  $ON, O'N'$  perpendicular to  $CD$ .

Also draw  $OK, OL$  perpendicular to  $OM$  and  $ON$ :

then  $OM \cdot AB + ON \cdot CD = O'M' \cdot AB + O'N' \cdot CD$ .

Hence  $OK \cdot AB = OL \cdot CD$ .

Therefore  $OL : OK$  in a constant ratio.

Hence the locus is a straight line.

125. By Art. 241 the sum or difference of the tangents is proportional to the distance between the ordinates of the points where the circles touch the curve according as the point does or does not lie between those ordinates.

126. If  $SY, S'Y'$  be the perpendiculars from the focus on the tangent at  $P$ ,  $CY'$  is parallel to  $SP$ ; and, if  $DK$  is the perpendicular on  $CY'$ ,

$$DK : CD = SY : SP = BC :: CD; \\ \therefore DK = BC.$$

127. If  $V$  and  $T$  are contiguous corners of the parallelogram formed by the tangents, and if  $CV$  and  $CT$  meet in  $E$  and  $F$  the sides of the parallelogram formed by the points of contact,

$$CE \cdot CV = CP^2 \text{ and } CF \cdot CT = CD^2; \\ \therefore (CE \cdot CF) (CV \cdot CT) = (CP \cdot CD)^2.$$

128. Taking the figure of Art. 12, let  $TL, TM, TN$  be drawn perpendicular to  $SF, SP, FF'$ , and let the circle intersect  $FF'$  in  $G$  and  $G'$ ;

$$\text{then } TG : TN :: TL : TN :: SM : TN \\ :: SA : AX,$$

$\therefore TG$  is parallel to an asymptote.

129. For  $TS$  bisects  $QSq$ , Art. 12, and  $FS$  bisects the outer angle, Art. 5.

130. Let the chord  $Qq$ , normal at  $q$ , meet the directrix in  $F$ , and let  $T$  be the pole of  $Qq$ ; then  $S$  being the pole of the directrix,  $ST$  is the polar of  $F$ , and therefore  $T$  being a point in the directrix,  $TSF$  is a right angle.

Taking  $V$  as the middle point of  $Qq$ , let  $FS$  meet in  $L$  the polar of  $V$ , which is parallel to  $Qq$ .

$$\begin{aligned}
 \text{Then } LSQ &= TSQ - TSL = TSq - TSF \\
 &= FSq = SqV - SFq \\
 &= PqV - STq, \because T, S, q, V \text{ are concyclic,} \\
 &= PqV - QTV, \text{ Art. 39,} \\
 &= PVq - QTV = QVN - QTV \\
 &= TQV = LTQ, \because TL, QV \text{ are parallel.} \\
 \therefore L, T, S, Q &\text{ are concyclic, and} \\
 TQL &= TSL = 90^\circ.
 \end{aligned}$$

131. (1) If the plane through the axis and the given point  $P$  intersects the cone in  $VA, VB$ , describe a circle passing through  $P$  and touching  $VA, VB$ ; then if  $APB$  is drawn touching the circle,  $AB$  is the axis of the section of which  $P$  is a focus.

(2) Produce  $VP$  to  $Q$  making  $PQ = PV$ , and in the plane above mentioned, draw  $VK$  parallel to  $VB$ , meeting  $VA$  in  $K$ , and  $VL$  parallel to  $VA$ , meeting  $VB$  in  $L$ ;

Then  $APL$  is the axis of a conic of which  $P$  is the centre.

132. If  $S, S'$  are the foci, and  $X$  the foot of the directrix,  $VS'S$  is a straight line, and  $XSS'$  is an isosceles triangle.

Taking  $A$  and  $A'$  as the corresponding vertices, draw  $AL$  and  $A'L'$  parallel to  $SS'$ , meeting  $AS'$  in  $L$  and  $AS$  in  $L'$ .

The latera recta are in the ratio of  $VS$  to  $VS'$ , and  $VS : AS = A'L' : AL'$  and  $VS' : A'S' = AL : A'L$ .

Now  $AX = LX$ ,  $A'X = L'X$ , and  $A'L = AL'$ ;  
 but  $VS \cdot AL' = AS \cdot A'L$  and  $VS' \cdot A'L = A'S' \cdot AL$ ;  
 $\therefore VS : VS' = AS \cdot A'X : A'S'AX$ .





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